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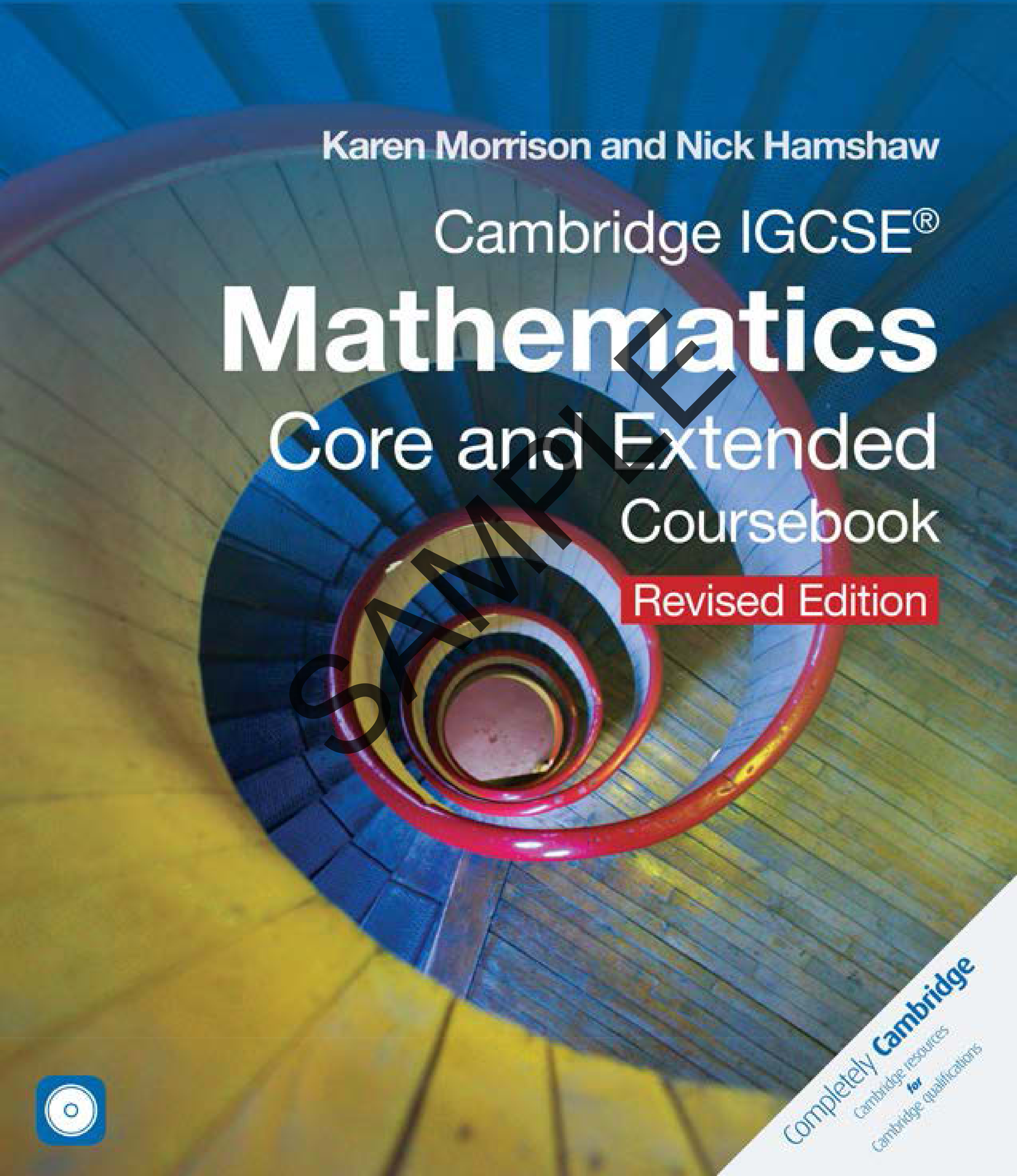
Karen Morrison and Nick Hamshaw

Cambridge IGCSE®
Mathematics
Core and Extended
Coursebook

Revised Edition



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Revised Edition

Cambridge University Press

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Contents

Unit 4

Unit 5

Unit 6

Introduction

This highly illustrated coursebook covers the complete *Cambridge IGCSE Mathematics* (0580) syllabus. Core and Extended material is combined in one book, offering a one-stop-shop for all students and teachers. Useful hints are included in the margins for students needing more support, leaving the narrative clear and to the point. The material required for the Extended course is clearly marked using colour panels; Extended students are given access to the parts of the Core syllabus they need without having to use an additional book.

REWIND

You learned how to plot lines from equations in chapter 10. ◀

FAST FORWARD

You will learn much more about sets in chapter 9. For now, just think of a set as a list of numbers or other items that are often placed inside curly brackets. ▶

The coursebook has been written with a clear progression from start to finish, with some later chapters requiring knowledge learned in earlier chapters. There are useful signposts throughout the coursebook that link the content of the chapters, allowing the individual to follow their own course through the book: where the content in one chapter might require knowledge from a previous chapter, a comment is included in a 'Rewind' box; and where content will be covered in more detail later on in the coursebook, a comment is included in a 'Fast forward' box. Examples of both are included here.

Worked examples are used throughout to demonstrate each method using typical workings and thought processes. These present the methods to the students in a practical and easy-to-follow way that minimises the need for lengthy explanations.

There is plenty of practice offered via 'drill' exercises throughout each chapter. The exercises are progressive questions that allow the student to practise methods that have just been introduced. At the end of each chapter there are 'Exam-style' questions and 'Past paper' questions. The 'Exam-style' questions have been written by the authors in the *style* of examination questions. The 'Past paper' questions are *real* questions taken from past exam papers. Both these end of chapter questions typically reflect the kinds of short answer questions that you may face in examinations though you will find some more structured ones in there as well. The answers to *all* of these questions are supplied at the back of the book, allowing self- and/or class- assessment. You can assess your progress as they go along, choosing to do more or less practise as required.

The suggested progression through the coursebook is for Units 1-3 to be covered in the first year of both courses, and units 4-6 to be covered in the second year of both courses. On this basis, there is mixed exam practice at the end of Unit 3 and the end of Unit 6. This is however, only a **suggested** structure and the course can be taught in various different ways; the signposting throughout the coursebook means that it can be used alongside any order of teaching. The end of unit questions represent the longer answer 'structured' examination questions and will use a combination of methods from across all relevant chapters. As with the end of chapter questions, these are a mixture of 'Exam-style' and 'Past paper' questions. The answers to these questions are on the Teacher's resource so that they can be used in classroom tests or for homework, if desired.

The coursebook also comes with a glossary to provide a definition for important / tricky terms.

Helpful guides in the margin of the book include:

Remember 'coefficient' is the *number* in the term.

Clues: these are general comments to remind you of important or key information that is useful to tackle an exercise, or simply useful to know. They often provide extra information or support in potentially tricky topics.

Tip

It is essential that you remember to work out *both* unknowns. Every pair of simultaneous linear equations will have a pair of solutions.

Tip: They cover common pitfalls based on the **authors'** experiences of their students, and give you things to be wary of or to remember.

The accompanying student CD-ROM at the back of the coursebook includes:

- A 'coverage grid' to map the contents of the syllabus to the topics and chapters in the coursebook.
- A 'Calculator support' chapter. This chapter covers the main uses of calculators that students seem to struggle with, and includes some worksheets to provide practice at using your calculator in these situations.
- Revision:
  - Core revision worksheets (and answers) provide extra exercises for each chapter of the book. These worksheets contain only content from the Core syllabus.
  - Extended revision worksheets (and answers) provide extra exercises for each chapter of the book. These worksheets contain the **same** questions as the Core worksheets, **in addition to** some more challenging questions, *and* questions to cover content unique to the Extended syllabus. Students are encouraged to do some (if not all) of the 'Core' questions on these worksheets, as well as the Extended ones (shaded) in order to fully revise the course. If time is limited, you might find it easier to pick two or three 'Core' questions to do before moving on to the 'Extended' questions.
  - Quick revision is a set of interactive questions in the form of multiple choice, drag and drop, or hide and reveal. They are quick-fire questions to test yourself in a different medium to pen and paper, and to get you thinking on the spot. They cover the Core content, with only a few additional screens being specific to the Extended course. There is at least one activity for each chapter. **Students are recommended to use the revision worksheets for a more comprehensive revision exercise.**
  - Worked solutions are interactive hide and reveal screens showing worked solutions to **some** of the end of chapter exam practice questions. Some of these will be 'Exam-style' and some will be 'Past paper' questions but all will be taken from the end of a chapter. There will be at least one for each chapter. The screen includes the question, and the answer, but also includes a series of 'Clue' or 'Tip' boxes. The 'Clue' boxes can be clicked on to reveal a clue to help the student if they are struggling with how to approach the question. The 'Tip' boxes contain tips relating to the exam, just like the 'Tip' boxes in the coursebook.

Also in the *Cambridge IGCSE Mathematics* series are two Practice Books – one for Core and one for Extended – to offer students targeted practice. These follow the chapters and topics of the coursebook to offer additional exercises for those who want more practice. These too, include 'Clues' and 'Tips' to help with tricky topics. There is also a Teacher's resource CD-ROM to offer support and advice

Acknowledgements

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1 Reviewing number concepts

Key words

- Natural number
- Integer
- Prime number
- Symbol
- Multiple
- Factor
- Composite numbers
- Prime factor
- Square root
- Cube
- Directed numbers
- BODMAS

In this chapter you will learn how to:

- identify and classify different types of numbers
- find common factors and common multiples of numbers
- write numbers as products of their prime factors
- calculate squares, square roots, cubes and cube roots of numbers
- work with integers used in real-life situations
- revise the basic rules for operating with numbers
- perform basic calculations using mental methods and with a calculator.

This statue is a replica of a 22 000-year-old bone found in the Congo. The real bone is only 10 cm long and it is carved with groups of notches that represent numbers. One column lists the prime numbers from 10 to 20. It is one of the earliest examples of a number system using tallies.

Our modern number system is called the Hindu-Arabic system because it was developed by Hindus and spread by Arab traders who brought it with them when they moved to different places in the world. The Hindu-Arabic system is decimal. This means it uses place value based on powers of ten. Any number at all, including decimals and fractions, can be written using place value and the digits from 0 to 9.

1.1 Different types of numbers

Make sure you know the correct mathematical words for the types of numbers in the table.

You should already be familiar with most of the concepts in this chapter. It is included here so that you can revise the concepts and check that you remember them.

FAST FORWARD

You will learn about the difference between rational and irrational numbers in chapter 9. ▶

Number	Definition	Example
Natural number	Any whole number from 1 to infinity, sometimes called 'counting numbers'. 0 is not included.	1, 2, 3, 4, 5, . . .
Odd number	A whole number that cannot be divided exactly by 2.	1, 3, 5, 7, . . .
Even number	A whole number that can be divided exactly by 2.	2, 4, 6, 8, . . .
Integer	Any of the negative and positive whole numbers, including zero.	. . . −3, −2, −1, 0, 1, 2, 3, . . .
Prime number	A whole number greater than 1 which has only two factors: the number itself and 1.	2, 3, 5, 7, 11, . . .
Square number	The product obtained when an integer is multiplied by itself.	1, 4, 9, 16, . . .
Fraction	A number representing parts of a whole number, can be written as a common (vulgar) fraction in the form of $\frac{a}{b}$ or as a decimal using the decimal point.	$\frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{8}, \frac{13}{3}, 2\frac{1}{2}$ 0.5, 0.2, 0.08, 1.7

Find the 'product' means 'multiply'. So, the product of 3 and 4 is 12, i.e. 3 × 4 = 12.

Exercise 1.1

FAST FORWARD

You will learn much more about sets in chapter 9. For now, just think of a set as a list of numbers or other items that are often placed inside curly brackets. ▶

1 Here is a set of numbers: {−4, −1, 0, $\frac{1}{2}$, 0.75, 3, 4, 6, 11, 16, 19, 25}

List the numbers from this set that are:

(a) natural numbers **(b)** even numbers **(c)** odd numbers
(d) integers **(e)** negative integers **(f)** fractions
(g) square numbers **(h)** prime numbers **(i)** neither square nor prime.

2 List:

(a) the next four odd numbers after 107
(b) four consecutive even numbers between 2008 and 2030
(c) all odd numbers between 993 and 1007
(d) the first five square numbers
(e) four decimal fractions that are smaller than 0.5
(f) four vulgar fractions that are greater than $\frac{1}{2}$ but smaller than $\frac{3}{4}$.

3 State whether the following will be odd or even:

(a) the sum of two odd numbers
(b) the sum of two even numbers
(c) the sum of an odd and an even number
(d) the square of an odd number
(e) the square of an even number
(f) an odd number multiplied by an even number.

Remember that a 'sum' is the result of an addition. The term is often used for *any* calculation in early mathematics but its meaning is very specific at this level.

Living maths

4 There are many other types of numbers. Find out what these numbers are and give an example of each.

(a) Perfect numbers.
(b) Palindromic numbers.
(c) Narcissistic numbers. (In other words, numbers that love themselves!)

Using symbols to link numbers

Mathematicians use numbers and **symbols** to write mathematical information in the shortest, clearest way possible.

You have used the operation symbols +, −, × and ÷ since you started school. Now you will also use the symbols given in the margin below to write mathematical statements.

Exercise 1.2

$=$ is equal to
$\neq$ is not equal to
$\approx$ is approximately equal to
$<$ is less than
$\leq$ is less than or equal to
$>$ is greater than
$\geq$ is greater than or equal to
$\therefore$ therefore
$\sqrt{\ }$ the square root of

1 Rewrite each of these statements using mathematical symbols.

(a) 19 is less than 45
(b) 12 plus 18 is equal to 30
(c) 0.5 is equal to $\frac{1}{2}$
(d) 0.8 is not equal to 8.0
(e) −34 is less than 2 times −16
(f) therefore the number x equals the square root of 72
(g) a number (x) is less than or equal to negative 45
(h) π is approximately equal to 3.14
(i) 5.1 is greater than 5.01
(j) the sum of 3 and 4 is not equal to the product of 3 and 4
(k) the difference between 12 and −12 is greater than 12
(l) the sum of −12 and −24 is less than 0
(m) the product of 12 and a number (x) is approximately −40

Remember that the 'difference' between two numbers is the result of a subtraction. The order of the subtraction matters.

2 Say whether these mathematical statements are true or false.

(a) $0.599 > 6.0$
(b) $5 \times 1999 \approx 10\,000$
(c) $8.1 = 8\frac{1}{10}$
(d) $6.2 + 4.3 = 4.3 + 6.2$
(e) $20 \times 9 \geq 21 \times 8$
(f) $6.0 = 6$
(g) $-12 > -4$
(h) $19.9 \leq 20$
(i) $1000 > 199 \times 5$
(j) $\sqrt{16} = 4$
(k) $35 \times 5 \times 2 \neq 350$
(l) $20 \div 4 = 5 \div 20$
(m) $20 - 4 \neq 4 - 20$
(n) $20 \times 4 \neq 4 \times 20$

3 Work with a partner.

(a) Look at the symbols used on the keys of your calculator. Say what each one means in words.
(b) List any symbols that you do not know. Try to find out what each one means.

1.2 Multiples and factors

You can think of the multiples of a number as the 'times table' for that number. For example, the multiples of 3 are $3 \times 1 = \mathbf{3}$, $3 \times 2 = \mathbf{6}$, $3 \times 3 = \mathbf{9}$ and so on.

Multiples

A **multiple** of a number is found when you multiply that number by a positive integer. The first multiple of any number is the number itself (the number multiplied by 1).

Worked example 1

(a) What are the first three multiples of 12?
(b) Is 300 a multiple of 12?

(a)	12, 24, 36	To find these multiply 12 by 1, 2 and then 3. $12 \times 1 = 12$ $12 \times 2 = 24$ $12 \times 3 = 36$
(b)	Yes, 300 is a multiple of 12.	To find out, divide 300 by 12. If it goes exactly, then 300 is a multiple of 12. $300 \div 12 = 25$

Exercise 1.3

1 List the first five multiples of:

(a) 2 **(b)** 3 **(c)** 5 **(d)** 8
(e) 9 **(f)** 10 **(g)** 12 **(h)** 100

2 Use a calculator to find and list the first ten multiples of:

(a) 29 **(b)** 44 **(c)** 75 **(d)** 114
(e) 299 **(f)** 350 **(g)** 1012 **(h)** 9123

3 List:

(a) the multiples of 4 between 29 and 53
(b) the multiples of 50 less than 400
(c) the multiples of 100 between 4000 and 5000.

4 Here are five numbers: 576, 396, 354, 792, 1164. Which of these are multiples of 12?

5 Which of the following numbers are not multiples of 27?
(a) 324 **(b)** 783 **(c)** 816 **(d)** 837 **(e)** 1116

The lowest common multiple (LCM)

The lowest common multiple of two or more numbers is the smallest number that is a multiple of all the given numbers.

Worked example 2

Find the lowest common multiple of 4 and 7.

M_4 = 4, 8, 12, 16, 20, 24, **<u>28</u>**, 32 M_7 = 7, 14, 21, **<u>28</u>**, 35, 42 LCM = 28	List several multiples of 4. (Note: M_4 means multiples of 4.) List several multiples of 7. Find the lowest number that appears in both sets. This is the LCM.

Exercise 1.4

FAST FORWARD
Later in this chapter you will see how prime factors can be used to find LCMs.

1 Find the LCM of:

(a) 2 and 5 **(b)** 8 and 10 **(c)** 6 and 4
(d) 3 and 9 **(e)** 35 and 55 **(f)** 6 and 11
(g) 2, 4 and 8 **(h)** 4, 5 and 6 **(i)** 6, 8 and 9
(j) 1, 3 and 7 **(k)** 4, 5 and 8 **(l)** 3, 4 and 18

2 Is it possible to find the highest common multiple of two or more numbers? Give a reason for your answer.

Factors

A **factor** is a number that divides exactly into another number with no remainder. For example, 2 is a factor of 16 because it goes into 16 exactly 8 times. 1 is a factor of every number. The largest factor of any number is the number itself.

F_{12} means the factors of 12.

To list the factors in numerical order go down the left side and then up the right side of the factor pairs. Remember not to repeat factors.

Worked example 3

Find the factors of:

(a) 12 **(b)** 25 **(c)** 110

(a)	F_{12} = 1, 2, 3, 4, 6, 12	Find pairs of numbers that multiply to give 12: 1 × 12 2 × 6 3 × 4 Write the factors in numerical order.
(b)	F_{25} = 1, 5, 25	1 × 25 5 × 5 Do not repeat the 5.
(c)	F_{110} = 1, 2, 5, 10, 11, 22, 55, 110	1 × 110 2 × 55 5 × 22 10 × 11

Exercise 1.5

1 List all the factors of:

(a) 4 (b) 5 (c) 8 (d) 11 (e) 18
(f) 12 (g) 35 (h) 40 (i) 57 (j) 90
(k) 100 (l) 132 (m) 160 (n) 153 (o) 360

2 Which number in each set is not a factor of the given number?

(a) 14 {1, 2, 4, 7, 14}
(b) 15 {1, 3, 5, 15, 45}
(c) 21 {1, 3, 7, 14, 21}
(d) 33 {1, 3, 11, 22, 33}
(e) 42 {3, 6, 7, 8, 14}

FAST FORWARD

Later in this chapter you will learn more about divisibility tests and how to use these to decide whether or not one number is a factor of another. ▶

3 State true or false in each case.

(a) 3 is a factor of 313 (b) 9 is a factor of 99
(c) 3 is a factor of 300 (d) 2 is a factor of 300
(e) 2 is a factor of 122 488 (f) 12 is a factor of 60
(g) 210 is a factor of 210 (h) 8 is a factor of 420

4 What is the smallest factor and the largest factor of any number?

The highest common factor (HCF)

The highest common factor of two or more numbers is the highest number that is a factor of all the given numbers.

Worked example 4

Find the HCF of 8 and 24.

F_8 = 1, 2, 4, 8 F_{24} = 1, 2, 3, 4, 6, 8, 12, 24 HCF = 8	List the factors of each number. Underline factors that appear in both sets. Pick out the highest underlined factor (HCF).

Exercise 1.6

FAST FORWARD

You will learn how to find HCFs by using prime factors later in the chapter. ▶

1 Find the HCF of each pair of numbers.

(a) 3 and 6 **(b)** 24 and 16 **(c)** 15 and 40 **(d)** 42 and 70
(e) 32 and 36 **(f)** 26 and 36 **(g)** 22 and 44 **(h)** 42 and 48

2 Find the HCF of each group of numbers.

(a) 3, 9 and 15 **(b)** 36, 63 and 84 **(c)** 22, 33 and 121

3 Not including the factor provided, find two numbers that have:

(a) an HCF of 2 **(b)** an HCF of 6

4 What is the HCF of two different prime numbers? Give a reason for your answer.

Word problems involving HCF usually involve splitting things into smaller pieces or arranging things in equal groups or rows.

Living maths

5 Simeon has two lengths of rope. One piece is 72 metres long and the other is 90 metres long. He wants to cut both lengths of rope into the longest pieces of equal length possible. How long should the pieces be?

6 Ms Sanchez has 40 canvases and 100 tubes of paint to give to the students in her art group. What is the largest number of students she can have if she gives each student an equal number of canvasses and an equal number of tubes of paint?

7 Indira has 300 blue beads, 750 red beads and 900 silver beads. She threads these beads to make wire bracelets. Each bracelet must have the same number and colour of beads. What is the maximum number of bracelets she can make with these beads?

1.3 Prime numbers

Prime numbers have exactly two factors: one and the number itself.

Composite numbers have more than two factors.

The number 1 has only one factor so it is not prime and it is not composite.

Finding prime numbers

Over 2000 years ago, a Greek mathematician called Eratosthenes made a simple tool for sorting out prime numbers. This tool is called the 'Sieve of Eratosthenes' and the figure on page 7 shows how it works for prime numbers up to 100.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Cross out 1, it is not prime.

Circle 2, then cross out other multiples of 2.

Circle 3, then cross out other multiples of 3.

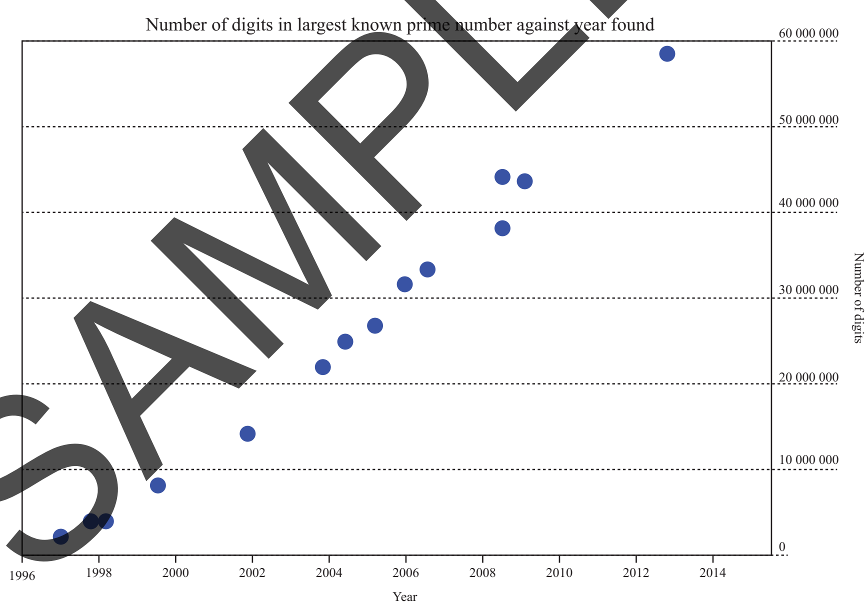
Circle the next available number then cross out all its multiples.

Repeat until all the numbers in the table are either circled or crossed out.

The circled numbers are the primes.

You should try to memorise which numbers between 1 and 100 are prime.

Other mathematicians over the years have developed ways of finding larger and larger prime numbers. Until 1955, the largest known prime number had less than 1000 digits. Since the 1970s and the invention of more and more powerful computers, more and more prime numbers have been found. The graph below shows the number of digits in the largest known primes since 1996.



Today anyone can join the Great Internet Mersenne Prime Search. This project links thousands of home computers to search continuously for larger and larger prime numbers while the computer processors have spare capacity.

Exercise 1.7

FAST FORWARD

A good knowledge of primes can help when factorising quadratics in chapter 10. ▶

1 Which is the only even prime number?

2 How many odd prime numbers are there less than 50?

3 **(a)** List the composite numbers greater than four, but less than 30.

(b) Try to write each composite number on your list as the sum of two prime numbers. For example: $6 = 3 + 3$ and $8 = 3 + 5$.

4 Twin primes are pairs of prime numbers that differ by two. List the twin prime pairs up to 100.

5 Is 149 a prime number? Explain how you decided.

> **Tip**
> Whilst super-prime numbers are interesting, they are not on the syllabus.

6 Super-prime numbers are prime numbers that stay prime each time you remove a digit (starting with the units). So, 59 is a super-prime because when you remove 9 you are left with 5, which is also prime. 239 is also a super-prime because when you remove 9 you are left with 23 which is prime, and when you remove 3 you are left with 2 which is prime.

(a) Find two three-digit super-prime numbers less than 400.
(b) Can you find a four-digit super-prime number less than 3000?
(c) Sondra's telephone number is the prime number 987-6413. Is her phone number a super-prime?

Prime factors

> Remember a product is the answer to a multiplication. So if you write a number as the product of its prime factors you are writing it using multiplication signs like this: $12 = 2 \times 2 \times 3$.

Prime factors are the factors of a number that are also prime numbers.

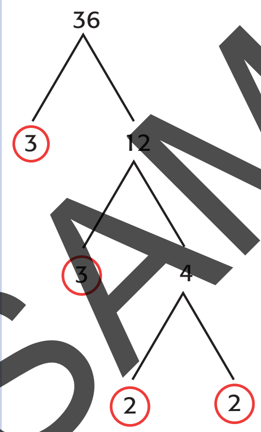
Every composite whole number can be broken down and written as the product of its prime factors. You can do this using tree diagrams or using division. Both methods are shown in worked example 5.

Worked example 5

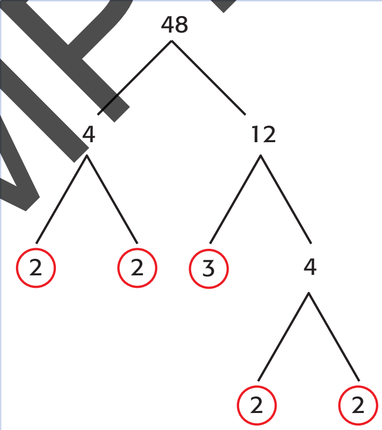
Write the following numbers as the product of prime factors.

(a) 36 **(b)** 48

Using a factor tree



$36 = 2 \times 2 \times 3 \times 3$



$48 = 2 \times 2 \times 2 \times 2 \times 3$

Write the number as two factors.

If a factor is a prime number, circle it.

If a factor is a composite number, split it into two factors.

Keep splitting until you end up with two primes.

Write the primes in ascending order with × signs.

> Prime numbers only have two factors: 1 and the number itself. As 1 is not a prime number, do not include it when expressing a number as a product of prime factors.

Using division

2	36
2	18
3	9
3	3
	1

$36 = 2 \times 2 \times 3 \times 3$

2	48
2	24
2	12
2	6
3	3
	1

$48 = 2 \times 2 \times 2 \times 2 \times 3$

Divide by the smallest prime number that will go into the number exactly.
Continue dividing, using the smallest prime number that will go into your new answer each time.
Stop when you reach 1.
Write the prime factors in ascending order with × signs.

> Choose the method that works best for you and stick to it. Always show your method when using prime factors.

Exercise 1.8

When you write your number as a product of primes, group all occurrences of the same prime number together.

1 Express the following numbers as the product of prime factors.

(a) 30	**(b)** 24	**(c)** 100	**(d)** 225	**(e)** 360
(f) 504	**(g)** 650	**(h)** 1125	**(i)** 756	**(j)** 9240

FAST FORWARD

You can also use prime factors to find the square and cube roots of numbers if you don't have a calculator. You will deal with this in more detail later in this chapter. ▶

Using prime factors to find the HCF and LCM

When you are working with larger numbers you can determine the HCF or LCM by expressing each number as a product of its prime factors.

Worked example 6

Find the HCF of 168 and 180.

$168 = \underline{2} \times \underline{2} \times 2 \times \underline{3} \times 7$ $180 = \underline{2} \times \underline{2} \times \underline{3} \times 3 \times 5$ $2 \times 2 \times 3 = 12$ HCF = 12	First express each number as a product of prime factors. Use tree diagrams or division to do this. Underline the factors *common* to both numbers. Multiply these out to find the HCF.

Worked example 7

Find the LCM of 72 and 120.

$72 = \underline{2} \times \underline{2} \times \underline{2} \times \underline{3} \times \underline{3}$ $120 = 2 \times 2 \times 2 \times 3 \times \underline{5}$ $2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$ LCM = 360	First express each number as a product of prime factors. Use tree diagrams or division to do this. Underline the *largest* set of multiples of each factor. List these and multiply them out to find the LCM.

Exercise 1.9

1 Find the HCF of these numbers by means of prime factors.

(a) 48 and 108	**(b)** 120 and 216	**(c)** 72 and 90	**(d)** 52 and 78
(e) 100 and 125	**(f)** 154 and 88	**(g)** 546 and 624	**(h)** 95 and 120

2 Use prime factorisation to determine the LCM of:

(a) 54 and 60	**(b)** 54 and 72	**(c)** 60 and 72	**(d)** 48 and 60
(e) 120 and 180	**(f)** 95 and 150	**(g)** 54 and 90	**(h)** 90 and 120

3 Determine both the HCF and LCM of the following numbers.

(a) 72 and 108	**(b)** 25 and 200	**(c)** 95 and 120	**(d)** 84 and 60

Living maths

Word problems involving LCM usually include repeating events. You may be asked how many items you need to 'have enough' or when something will happen again at the same time.

4 A radio station runs a phone-in competition for listeners. Every 30th caller gets a free airtime voucher and every 120th caller gets a free mobile phone. How many listeners must phone in before one receives both an airtime voucher *and* a free phone?

5 Lee runs round a track in 12 minutes. James runs round the same track in 18 minutes. If they start in the same place, at the same time, how many minutes will pass before they both cross the start line together again?

Divisibility tests to find factors easily

Sometimes you want to know if a smaller number will divide into a larger one with no remainder. In other words, is the larger number divisible by the smaller one?

These simple divisibility tests are useful for working this out:

A number is exactly divisible by:

2 if it ends with 0, 2, 4, 6 or 8 (in other words is even)
3 if the sum of its digits is a multiple of 3 (can be divided by 3)
4 if the last two digits can be divided by 4
5 if it ends with 0 or 5
6 if it is divisible by both 2 and 3
8 if the last three digits are divisible by 8
9 if the sum of the digits is a multiple of 9 (can be divided by 9)
10 if the number ends in 0.

There is no simple test for divisibility by 7, although multiples of 7 do have some interesting properties that you can investigate on the internet.

Tip

Divisibility tests are not part of the syllabus. They are just useful to know when you work with factors and prime numbers.

Exercise 1.10

23	65	92	10	104	70	500	21	64	798	1223

1 Look at the box of numbers above. Which of these numbers are:
(a) divisible by 5? **(b)** divisible by 8? **(c)** divisible by 3?

2 Say whether the following are true or false.
(a) 625 is divisible by 5 **(b)** 88 is divisible by 3
(c) 640 is divisible by 6 **(d)** 346 is divisible by 4
(e) 476 is divisible by 8 **(f)** 2340 is divisible by 9
(g) 2890 is divisible by 6 **(h)** 4562 is divisible by 3
(i) 40 090 is divisible by 5 **(j)** 123 456 is divisible by 9

3 Can \$34.07 be divided equally among:
(a) two people? **(b)** three people? **(c)** nine people?

4 A stadium has 202 008 seats. Can these be divided equally into:
(a) five blocks? **(b)** six blocks? **(c)** nine blocks?

5 **(a)** If a number is divisible by 12, what other numbers must it be divisible by?
(b) If a number is divisible by 36, what other numbers must it be divisible by?
(c) How could you test if a number is divisible by 12, 15 or 24?

1.4 Powers and roots

REWIND

In section 1.1 you learned that the product obtained when an integer is multiplied by itself is a square number. ◄

Square numbers and square roots

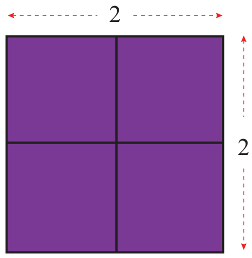
A number is squared when it is multiplied by itself. For example, the **square** of 5 is $5 \times 5 = 25$. The symbol for squared is ^{2}. So, 5×5 can also be written as 5^2.

The **square root** of a number is the number that was multiplied by itself to get the square number. The symbol for square root is $\sqrt{\ }$. You know that $25 = 5^2$, so $\sqrt{25} = 5$.

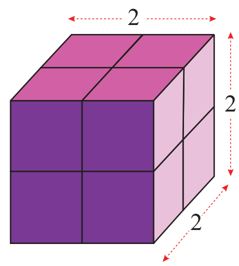
Cube numbers and cube roots

A number is cubed when it is multiplied by itself and then multiplied by itself again. For example, the **cube** of 2 is $2 \times 2 \times 2 = 8$. The symbol for cubed is ^{3}. So $2 \times 2 \times 2$ can also be written as 2^3.

The cube root of a number is the number that was multiplied by itself to get the cube number. The symbol for cube root is $\sqrt[3]{\ }$. You know that $8 = 2^3$, so $\sqrt[3]{8} = 2$.



a) Square numbers can be arranged to form a square shape. This is 2^2.



b) Cube numbers can be arranged to form a solid cube shape. This is 2^3.

Finding powers and roots

Not all calculators have exactly the same buttons. [$x^{\square}$] [x^y] and [^] all mean the same thing on different calculators.

You can use your calculator to square or cube numbers quickly using the [x^2] and [x^3] keys or the [$x^{\square}$] key. Use the [$\sqrt{\ }$] or [$\sqrt[3]{\ }$] keys to find the roots. If you don't have a calculator, you can use the product of prime factors method to find square and cube roots of numbers. Both methods are shown in the worked examples below.

Worked example 8

Use your calculator to find:

(a) 13^2 **(b)** 5^3 **(c)** $\sqrt{324}$ **(d)** $\sqrt[3]{512}$

(a)	$13^2 = 169$	Enter [1] [3] [x^2] [=]
(b)	$5^3 = 125$	Enter [5] [x^3] [=]. If you do not have a [x^3] button then enter [5] [$x^{\square}$] [3] [=]; for this key you have to enter the power.
(c)	$\sqrt{324} = 18$	Enter [$\sqrt{\ }$] [3] [2] [4] [=]
(d)	$\sqrt[3]{512} = 8$	Enter [$\sqrt[3]{\ }$] [5] [1] [2] [=]

Worked example 9

If you do not have a calculator, you can write the integer as a product of primes and group the prime factors into pairs or threes. Look again at parts (c) and (d) of worked example 8:

(c) $\sqrt{324}$ **(d)** $\sqrt[3]{512}$

(c)	$324 = \underbrace{2\times2}_{2}\times\underbrace{3\times3}_{3}\times\underbrace{3\times3}_{3}$	Group the factors into pairs, and write down the square root of each pair.
	$2\times3\times3 = 18$ $\sqrt{324} = 18$	Multiply the roots together to give you the square root of 324.
(d)	$512 = \underbrace{2\times2\times2}_{2}\times\underbrace{2\times2\times2}_{2}\times\underbrace{2\times2\times2}_{2}$	Group the factors into threes, and write the cube root of each threesome.
	$2\times2\times2 = 8$ $\sqrt[3]{512} = 8$	Multiply together to get the cube root of 512.

Exercise 1.11

1 Calculate:

(a) 3^2 (b) 7^2 (c) 11^2 (d) 12^2 (e) 21^2
(f) 19^2 (g) 32^2 (h) 100^2 (i) 14^2 (j) 68^2

2 Calculate:

(a) 1^3 (b) 3^3 (c) 4^3 (d) 6^3 (e) 9^3
(f) 10^3 (g) 100^3 (h) 18^3 (i) 30^3 (j) 200^3

Learn the squares of all integers between 1 and 20 inclusive. You will need to recognise these quickly.

3 Find a value of x to make each of these statements true.

(a) $x \times x = 25$ (b) $x \times x \times x = 8$ (c) $x \times x = 121$
(d) $x \times x \times x = 729$ (e) $x \times x = 324$ (f) $x \times x = 400$
(g) $x \times x \times x = 8000$ (h) $x \times x = 225$ (i) $x \times x \times x = 1$
(j) $\sqrt{x} = 9$ (k) $\sqrt{1} = x$ (l) $\sqrt{x} = 81$
(m) $\sqrt[3]{x} = 2$ (n) $\sqrt[3]{x} = 1$ (o) $\sqrt[3]{64} = x$

4 Use a calculator to find the following roots.

(a) $\sqrt{9}$ (b) $\sqrt{64}$ (c) $\sqrt{1}$ (d) $\sqrt{4}$ (e) $\sqrt{100}$
(f) $\sqrt{0}$ (g) $\sqrt{81}$ (h) $\sqrt{400}$ (i) $\sqrt{1296}$ (j) $\sqrt{1764}$
(k) $\sqrt[3]{8}$ (l) $\sqrt[3]{1}$ (m) $\sqrt[3]{27}$ (n) $\sqrt[3]{64}$ (o) $\sqrt[3]{1000}$
(p) $\sqrt[3]{216}$ (q) $\sqrt[3]{512}$ (r) $\sqrt[3]{729}$ (s) $\sqrt[3]{1728}$ (t) $\sqrt[3]{5832}$

5 Use the product of prime factors given below to find the square root of each number. Show your working.

(a) $324 = 2 \times 2 \times 3 \times 3 \times 3 \times 3$ (b) $225 = 3 \times 3 \times 5 \times 5$
(c) $784 = 2 \times 2 \times 2 \times 2 \times 7 \times 7$ (d) $2025 = 3 \times 3 \times 3 \times 3 \times 5 \times 5$
(e) $19\,600 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 7 \times 7$ (f) $250\,000 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

6 Use the product of prime factors to find the cube root of each number. Show your working.

(a) $27 = 3 \times 3 \times 3$ (b) $729 = 3 \times 3 \times 3 \times 3 \times 3 \times 3$
(c) $2197 = 13 \times 13 \times 13$ (d) $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$
(e) $15\,625 = 5 \times 5 \times 5 \times 5 \times 5 \times 5$
(f) $32\,768 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

Brackets act as grouping symbols. Work out any calculations inside brackets before doing the calculations outside the brackets.

Root signs work in the same way as a bracket. If you have $\sqrt{25+9}$, you must add 25 and 9 before finding the root.

7 Calculate:

(a) $(\sqrt{25})^2$ (b) $(\sqrt{49})^2$ (c) $(\sqrt[3]{64})^3$ (d) $(\sqrt[3]{32})^3$
(e) $\sqrt{9} + \sqrt{16}$ (f) $\sqrt{9+16}$ (g) $\sqrt{36} + \sqrt{64}$ (h) $\sqrt{36+64}$
(i) $\sqrt{100-36}$ (j) $\sqrt{100} - \sqrt{36}$ (k) $\sqrt{25} \times \sqrt{4}$ (l) $\sqrt{25 \times 4}$
(m) $\sqrt{9 \times 4}$ (n) $\sqrt{9} \times \sqrt{4}$ (o) $\sqrt{\frac{36}{4}}$ (p) $\frac{\sqrt{36}}{4}$

8 Find the length of the edge of a cube with a volume of:

(a) 1000 cm^3 (b) $19\,683\text{ cm}^3$ (c) $68\,921\text{ mm}^3$ (d) $64\,000\text{ cm}^3$

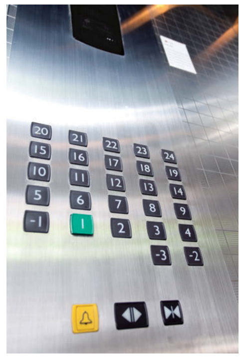
9 If the symbol * means 'add the square of the first number to the cube of the second number', calculate:

(a) 2 * 3 (b) 3 * 2 (c) 1 * 4 (d) 4 * 1 (e) 2 * 4
(f) 4 * 2 (g) 1 * 9 (h) 9 * 1 (i) 5 * 2 (j) 2 * 5

1.5 Working with directed numbers

Once a direction is chosen to be positive, the opposite direction is taken to be negative. So:

- if up is positive, down is negative
- if right is positive, left is negative
- if north is positive, south is negative
- if above 0 is positive, below 0 is negative.



A negative sign is used to indicate that values are less than zero. For example, on a thermometer, on a bank statement or in an elevator.

When you use numbers to represent real-life situations like temperatures, altitude, depth below sea level, profit or loss and directions (on a grid), you sometimes need to use the negative sign to indicate the direction of the number. For example, a temperature of three degrees below zero can be shown as −3 °C. Numbers like these, which have direction, are called **directed numbers**. So if a point 25 m above sea level is at +25 m, then a point 25 m below sea level is at −25 m.

Exercise 1.12

1 Express each of these situations using a directed number.

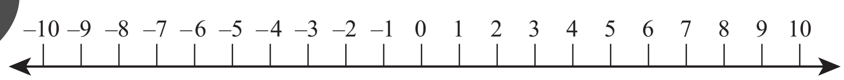
(a) a profit of \$100
(b) 25 km below sea level
(c) a drop of 10 marks
(d) a gain of 2 kg
(e) a loss of 1.5 kg
(f) 8000 m above sea level
(g) a temperature of 10 °C below zero
(h) a fall of 24 m
(i) a debt of \$2000
(j) an increase of \$250
(k) a time two hours behind GMT
(l) a height of 400 m
(m) a bank balance of \$450.00

Comparing and ordering directed numbers

FAST FORWARD

You will use similar number lines when solving linear inequalities in chapter 14.

In mathematics, directed numbers are also known as integers. You can represent the set of integers on a number line like this:



The further to the right a number is on the number line, the greater its value.

Exercise 1.13

It is important that you understand how to work with directed numbers early in your IGCSE course. Many topics depend upon them!

1 Copy the numbers and fill in < or > to make a true statement.

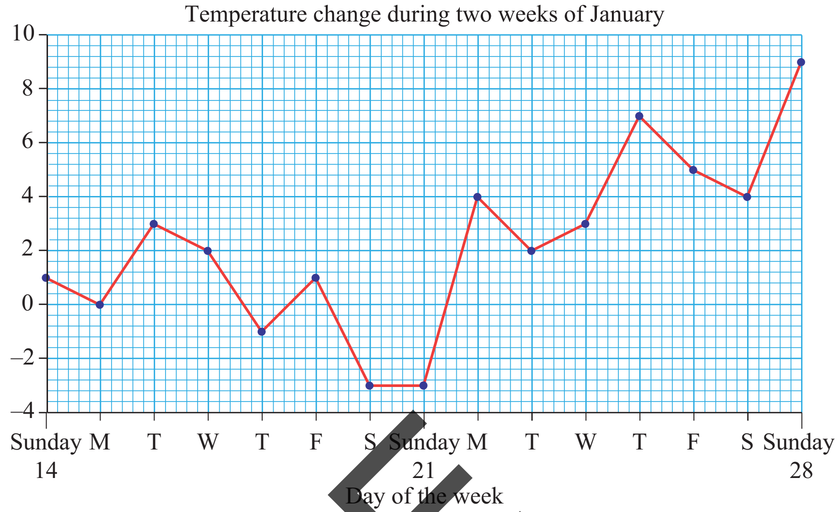
(a) 2 □ 8	**(b)** 4 □ 9	**(c)** 12 □ 3
(d) 6 □ −4	**(e)** −7 □ 4	**(f)** −2 □ 4
(g) −2 □ −11	**(h)** −12 □ −20	**(i)** −8 □ 0
(j) −2 □ 2	**(k)** −12 □ −4	**(l)** −32 □ −3
(m) 0 □ −3	**(n)** −3 □ 11	**(o)** 12 □ −89

2 Arrange each set of numbers in ascending order.

(a) −8, 7, 10, −1, −12
(b) 4, −3, −4, −10, 9, −8
(c) −11, −5, −7, 7, 0, −12
(d) −94, −50, −83, −90, 0

Living maths

3 Study the temperature graph carefully.



(a) What was the temperature on Sunday 14 January?
(b) By how much did the temperature drop from Sunday 14 to Monday 15?
(c) What was the lowest temperature recorded?
(d) What is the difference between the highest and lowest temperatures?
(e) On Monday 29 January the temperature changed by −12 degrees. What was the temperature on that day?

The difference between the highest and lowest temperature is also called the *range* of temperatures.

4 Matt has a bank balance of \$45.50. He deposits \$15.00 and then withdraws \$32.00. What is his new balance?

5 Mr Singh's bank account is \$420 overdrawn.

(a) Express this as a directed number.
(b) How much money will he need to deposit to get his account to have a balance of \$500?
(c) He deposits \$200. What will his new balance be?

6 A diver 27 m below the surface of the water rises 16 m. At what depth is she then?

7 On a cold day in New York, the temperature at 6 a.m. was −5 °C. By noon, the temperature had risen to 8 °C. By 7 p.m. the temperature had dropped by 11 °C from its value at noon. What was the temperature at 7 p.m.?

8 Local time in Abu Dhabi is four hours ahead of Greenwich Mean Time. Local time in Rio de Janeiro is three hours behind Greenwich Mean Time.

(a) If it is 4 p.m. at Greenwich, what time is it in Abu Dhabi?
(b) If it is 3 a.m. in Greenwich, what time is it in Rio de Janeiro?
(c) If it is 3 p.m. in Rio de Janeiro, what time is it in Abu Dhabi?
(d) If it is 8 a.m. in Abu Dhabi, what time is it in Rio de Janeiro?

1.6 Order of operations

At this level of mathematics you are expected to do more complicated calculations involving more than one operation (+, −, × and ÷). When you are carrying out more complicated calculations you have to follow a sequence of rules so that there is no confusion about what operations you should do first. The rules governing the order of operations are:

- complete operations in grouping symbols first (see page 15)
- do division and multiplication next, working from left to right
- do addition and subtractions last, working from left to right.

Many people use the letters **BODMAS** to remember the order of operations. The letters stand for:

Brackets

Of (Sometimes, 'I' for 'indices' is used instead of 'O' for 'of')

Divide **M**ultiply

Add **S**ubtract

BODMAS indicates that indices (powers) are considered after brackets but before all other operations.

Grouping symbols

The most common grouping symbols in mathematics are brackets. Here are some examples of the different kinds of brackets used in mathematics:

$(4 + 9) \times (10 \div 2)$

$[2(4 + 9) - 4(3) - 12]$

$\{2 - [4(2 - 7) - 4(3 + 8)] - 2 \times 8\}$

When you have more than one set of brackets in a calculation, you work out the innermost set first.

Other symbols used to group operations are:

- fraction bars, e.g. $\frac{5-12}{3-8}$
- root signs, such as square roots and cube roots, e.g. $\sqrt{9+16}$
- powers, e.g. 5^2 or 4^3

Worked example 10

Simplify:

(a) $7 \times (3 + 4)$ **(b)** $(10 - 4) \times (4 + 9)$ **(c)** $45 - [20 \times (4 - 3)]$

(a) $7 \times 7 = 49$

(b) $6 \times 13 = 78$

(c) $45 - [20 \times 1] = 45 - 20$
$= 25$

Worked example 11

Calculate:

(a) $3 + 8^2$ **(b)** $\frac{4+28}{17-9}$ **(c)** $\sqrt{36 \div 4} + \sqrt{100 - 36}$

(a) $3 + (8 \times 8)$
$= 3 + 64$
$= 67$

(b) $(4 + 28) \div (17 - 9)$
$= 32 \div 8$
$= 4$

(c) $\sqrt{36 \div 4} + \sqrt{100 - 36}$
$= \sqrt{9} + \sqrt{64}$
$= 3 + 8$
$= 11$

Exercise 1.14

1 Calculate. Show the steps in your working.

(a) $(4 + 7) \times 3$
(b) $(20 - 4) \div 4$
(c) $50 \div (20 + 5)$
(d) $6 \times (2 + 9)$
(e) $(4 + 7) \times 4$
(f) $(100 - 40) \times 3$
(g) $16 + (25 \div 5)$
(h) $19 - (12 + 2)$
(i) $40 \div (12 - 4)$
(j) $100 \div (4 + 16)$
(k) $121 \div (33 \div 3)$
(l) $15 \times (15 - 15)$

2 Calculate:

(a) $(4 + 8) \times (16 - 7)$
(b) $(12 - 4) \times (6 + 3)$
(c) $(9 + 4) - (4 + 6)$
(d) $(33 + 17) \div (10 - 5)$
(e) $(4 \times 2) + (8 \times 3)$
(f) $(9 \times 7) \div (27 - 20)$
(g) $(105 - 85) \div (16 \div 4)$
(h) $(12 + 13) \div 5^2$
(i) $(56 - 6^2) \times (4 + 3)$

A bracket 'type' is always twinned with another bracket of the same type/shape. This helps mathematicians to understand the order of calculations even more easily.

3 Simplify. Remember to work from the innermost grouping symbols to the outermost.

(a) $4 + [12 - (8 - 5)]$
(b) $6 + [2 - (2 \times 0)]$
(c) $8 + [60 - (2 + 8)]$
(d) $200 - [(4 + 12) - (6 + 2)]$
(e) $200\{100 - [4 \times (2 + 8)]\}$
(f) $\{6 + [5 \times (2 + 30)]\} \times 10$
(g) $[(30 + 12) - (7 + 9)] \times 10$
(h) $6 \times [(20 \div 4) - (6 - 3) + 2]$
(i) $1000 - [6 \times (4 + 20) - 4 \times (3 + 0)]$

4 Calculate:

(a) $6 + 72$
(b) $29 - 23$
(c) 8×42
(d) $20 - 4 \div 2$
(e) $\dfrac{31 - 10}{14 - 7}$
(f) $\dfrac{100 - 40}{5 \times 4}$
(g) $\sqrt{100 - 36}$
(h) $\sqrt{8 + 8}$
(i) $\sqrt{90 - 9}$

5 Insert brackets into the following calculations to make them true.

(a) $3 \times 4 + 6 = 30$
(b) $25 - 15 \times 9 = 90$
(c) $40 - 10 \times 3 = 90$
(d) $14 - 9 \times 2 = 10$
(e) $12 + 3 \div 5 = 3$
(f) $19 - 9 \times 15 = 150$
(g) $10 + 10 \div 6 - 2 = 5$
(h) $3 + 8 \times 15 - 9 = 66$
(i) $9 - 4 \times 7 + 2 = 45$
(j) $10 - 4 \times 5 = 30$
(k) $6 \div 3 + 3 \times 5 = 5$
(l) $15 - 6 \div 2 = 12$
(m) $1 + 4 \times 20 \div 5 = 20$
(n) $8 + 5 - 3 \times 2 = 20$
(o) $36 \div 3 \times 3 - 3 = 6$
(p) $3 \times 4 - 2 \div 6 = 1$
(q) $40 \div 4 + 1 = 11$
(r) $6 + 2 \times 8 + 2 = 24$

FAST FORWARD

You will apply the order of operation rules to fractions, decimals and algebraic expressions as you progress through the course.

Working in the correct order

Now that you know what to do with grouping symbols, you are going to apply the rules for order of operations to perform calculations with numbers.

Exercise 1.15

1 Simplify. Show the steps in your working.

(a) $5 \times 10 + 3$
(b) $5 \times (10 + 3)$
(c) $2 + 10 \times 3$
(d) $(2 + 10) \times 3$
(e) $23 + 7 \times 2$
(f) $6 \times 2 \div (3 + 3)$
(g) $\dfrac{15 - 5}{2 \times 5}$
(h) $(17 + 1) \div 9 + 2$
(i) $\dfrac{16 - 4}{4 - 1}$
(j) $17 + 3 \times 21$
(k) $48 - (2 + 3) \times 2$
(l) $12 \times 4 - 4 \times 8$
(m) $15 + 30 \div 3 + 6$
(n) $20 - 6 \div 3 + 3$
(o) $10 - 4 \times 2 \div 2$

2 Simplify:

(a) $18 - 4 \times 2 - 3$
(b) $14 - (21 \div 3)$
(c) $24 \div 8 \times (6 - 5)$
(d) $42 \div 6 - 3 - 4$
(e) $5 + 36 \div 6 - 8$
(f) $(8 + 3) \times (30 \div 3) \div 11$

3 State whether the following are true or false.

(a) $(1 + 4) \times 20 + 5 = 1 + (4 \times 20) + 5$
(b) $6 \times (4 + 2) \times 3 > (6 \times 4) \div 2 \times 3$
(c) $8 + (5 - 3) \times 2 < 8 + 5 - (3 \times 2)$
(d) $100 + 10 \div 10 > (100 + 10) \div 10$

4 Place the given numbers in the correct spaces to make a correct number sentence.

(a) 0, 2, 5, 10 □ − □ ÷ □ = □
(b) 9, 11, 13, 18 □ − □ ÷ □ = □
(c) 1, 3, 8, 14, 16 □ ÷ (□ − □) − □ = □
(d) 4, 5, 6, 9, 12 (□ + □) − (□ − □) = □

Using your calculator

In this section you will use your calculator to perform operations in the correct order. However, you will need to remember the *order of operations* rules and apply them throughout the book as you do more complicated examples using your calculator.

A calculator with algebraic logic will apply the rules for order of operations automatically. So, if you enter $2 + 3 \times 4$, your calculator will do the multiplication first and give you an answer of 14. (Check that your calculator does this!).

When the calculation contains brackets you must enter these to make sure your calculator does the grouped sections first.

Experiment with your calculator by making several calculations with and without brackets. For example: $3 \times 2 + 6$ and $3 \times (2 + 6)$. Do you understand why these are different?

Your calculator might only have one type of bracket [( ] and [ )]. If there are two different shaped brackets in the calculation (such as $[4 \times (2 - 3)]$, enter the calculator bracket symbol for each type.

Worked example 12

Use a calculator to find:

(a) $3 + 2 \times 9$ **(b)** $(3 + 8) \times 4$ **(c)** $(3 \times 8 - 4) - (2 \times 5 + 1)$

(a)	21	Enter [3] [+] [2] [×] [9] [=]
(b)	44	Enter [(] [3] [+] [8] [)] [×] [4] [=]
(c)	9	Enter [(] [3] [×] [8] [−] [4] [)] [−] [(] [2] [×] [5] [+] [1] [)] [=]

Exercise 1.16

Some calculators have two '–' buttons: [−] and [(−)]. The first means 'subtract' and is used to subtract one number from another. The second means 'make negative'. Experiment with the buttons and make sure that your calculator is doing what you expect it to do!

1 Use a calculator to find the correct answer.

(a) $10 - 4 \times 5$ (b) $12 + 6 \div 7 - 4$
(c) $3 + 4 \times 5 - 10$ (d) $18 \div 3 \times 5 - 3 + 2$
(e) $5 - 3 \times 8 - 6 \div 2$ (f) $7 + 3 \div 4 + 1$
(g) $(1 + 4) \times 20 \div 5$ (h) $36 \div 6 \times (3 - 3)$
(i) $(8 + 8) - 6 \times 2$ (j) $100 - 30 \times (4 - 3)$
(k) $24 \div (7 + 5) \times 6$ (l) $[(60 - 40) - (53 - 43)] \times 2$
(m) $[(12 + 6) \div 9] \times 4$ (n) $[100 \div (4 + 16)] \times 3$
(o) $4 \times [25 \div (12 - 7)]$

2 Use your calculator to check whether the following answers are correct. If the answer is incorrect, work out the correct answer.

(a) $12 \times 4 + 76 = 124$ (b) $8 + 75 \times 8 = 698$
(c) $12 \times 18 - 4 \times 23 = 124$ (d) $(16 \div 4) \times (7 + 3 \times 4) = 76$
(e) $(82 - 36) \times (2 + 6) = 16$ (f) $(3 \times 7 - 4) - (4 + 6 \div 2) = 12$

3 Each * represents a missing operation. Work out what it is.

(a) 12 * (28 * 24) = 3 (b) 84 * 10 * 8 = 4
(c) 3 * 7(0.7 * 1.3) = 17 (d) 23 * 11 * 22 * 11 = 11
(e) 40 * 5 * (7 * 5) = 4 (f) 9 * 15 * (3 * 2) = 12

The more effectively you are able to use your calculator, the faster and more accurate your calculations are likely to be. If you have difficulty with this you will find advice and practice exercises on the CD-ROM.

4 Calculate:

(a) $\dfrac{7 \times \sqrt{16}}{2^3 + 7^2 - 1}$ (b) $\dfrac{5^2 \times \sqrt{4}}{1 + 6^2 - 12}$ (c) $\dfrac{2 + 3^2}{5^2 + 4 \times 10 - \sqrt{25}}$

(d) $\dfrac{6^2 - 11}{2(17 + 2 \times 4)}$ (e) $\dfrac{3^2 - 3}{2 \times \sqrt{81}}$ (f) $\dfrac{3^2 - 5 + 6}{\sqrt{4} \times 5}$

(g) $\dfrac{36 - 3 \times \sqrt{16}}{15 - 3^2 \div 3}$ (h) $\dfrac{-30 + [18 \div (3 - 12) + 24]}{5 - 8 - 3^2}$

5 Use a calculator to find the answer.

(a) $\dfrac{0.345}{1.34 + 4.2 \times 7}$ (b) $\dfrac{12.32 \times 0.0378}{\sqrt{16} + 8.05}$ (c) $\dfrac{\sqrt{16} \times 0.087}{2^2 - 5.098}$ (d) $\dfrac{19.23 \times 0.087}{2.45^2 - 1.03^2}$

FAST FORWARD

In this chapter you are only dealing with square and cube numbers, and the roots of square and cube numbers. When you work with indices and standard form in chapter 5, you will need to apply these skills and use your calculator effectively to solve problems involving any powers or roots. ▶

6 Use your calculator to evaluate.

(a) $\sqrt{64 \times 125}$ (b) $\sqrt{2^3 \times 3^2 \times 6}$

(c) $\sqrt[3]{8^2 + 19^2}$ (d) $\sqrt{41^2 - 36^2}$

(e) $\sqrt{3.2^2 - 1.17^3}$ (f) $\sqrt[3]{1.45^3 - 0.13^2}$

(g) $\dfrac{1}{4}\sqrt{\dfrac{1}{4} + \dfrac{1}{4} + \sqrt{\dfrac{1}{4}}}$ (h) $\sqrt[3]{2.75^2 - \frac{1}{2} \times 1.7^3}$

1.7 Rounding numbers

In many calculations, particularly with decimals, you will not need to find an exact answer. Instead, you will be asked to give an answer to a stated level of accuracy. For example, you may be asked to give an answer correct to 2 decimal places, or an answer correct to 3 significant figures.

To round a number to a given decimal place you look at the value of the digit to the right of the specified place. If it is 5 or greater, you round up; if it less than 5, you round down.

Worked example 13

Round 64.839906 to:

(a) the nearest whole number **(b)** 1 decimal place **(c)** 3 decimal places

(a)	$6\underline{4}.839906$ $64.\underline{8}39906$ = 65 (to nearest whole number)	4 is in the units place. The next digit is 8, so you will round up to get 5. To the nearest whole number.
(b)	$64.\underline{8}39906$ $64.8\underline{3}9906$ = 64.8 (1dp)	8 is in the first decimal place. The next digit is 3, so the 8 will remain unchanged. Correct to 1 decimal place.
(c)	$64.83\underline{9}906$ $64.839\underline{9}06$ = 64.840 (3dp)	9 is in the third decimal place. The next digit is 9, so you need to round up. When you round 9 up, you get 10, so carry one to the previous digit and write 0 in the place of the 9. Correct to 3 decimal places.

The first significant digit of a number is the first *non-zero* digit, when reading from left to right. The next digit is the second significant digit, the next the third significant and so on. All zeros *after* the first significant digit are considered significant.

To round to 3 significant figures, find the third significant digit and look at the value of the digit to the right of it. If it is 5 or greater, add one to the third significant digit and lose all of the other digits to the right. If it is less than 5, leave the third significant digit unchanged and lose all the other digits to the right as before. To round to a different number of significant figures, use the same method but find the appropriate significant digit to start with: the fourth for 4sf, the seventh for 7sf etc. If you are rounding to a whole number, write the appropriate number of zeros after the last significant digit as place holders to keep the number the same size.

Worked example 14

Round:

(a) 1.076 to 3 significant figures **(b)** 0.00736 to 1 significant figure

(a)	1.076 = 1.08 (3sf)	The third significant figure is the 7. The next digit is 6, so round 7 up to get 8. Correct to 3 significant figures.
(b)	0.00736 = 0.007 (1sf)	The first significant figure is the 7. The next digit is 3, so 7 will not change. Correct to 1 significant figure.

Exercise 1.17

Remember, the first significant digit in a number is the first *non-zero* digit, reading from left to right. Once you have read past the first non-zero digit, all zeros then become significant.

FAST FORWARD

You will use rounding to a given number of decimal places and significant figures in almost all of your work this year. You will also apply these skills to estimate answers. This is dealt with in more detail in chapter 5.

1 Round each number to 2 decimal places.

(a) 3.185 **(b)** 0.064 **(c)** 38.3456 **(d)** 2.149 **(e)** 0.999
(f) 0.0456 **(g)** 0.005 **(h)** 41.567 **(i)** 8.299 **(j)** 0.4236
(k) 0.062 **(l)** 0.009 **(m)** 3.016 **(n)** 12.0164 **(o)** 15.11579

2 Express each number correct to:

(i) 4 significant figures **(ii)** 3 significant figures **(iii)** 1 significant figure

(a) 4512 **(b)** 12 305 **(c)** 65 238 **(d)** 320.55
(e) 25.716 **(f)** 0.000765 **(g)** 1.0087 **(h)** 7.34876
(i) 0.00998 **(j)** 0.02814 **(k)** 31.0077 **(l)** 0.0064735

3 Change $2\frac{5}{9}$ to a decimal using your calculator. Express the answer correct to:

(a) 3 decimal places **(b)** 2 decimal places **(c)** 1 decimal place
(d) 3 significant figures **(e)** 2 significant figures **(f)** 1 significant figure

Summary

Do you know the following?

- Numbers can be classified as natural numbers, integers, prime numbers and square numbers.
- When you multiply an integer by itself you get a square number (x^2). If you multiply it by itself again you get a cube number (x^3).
- The number you multiply to get a square is called the square root and the number you multiply to get a cube is called the cube root. The symbol for square root is $\sqrt{\ }$. The symbol for cube root is $\sqrt[3]{\ }$.
- A multiple is obtained by multiplying a number by a natural number. The LCM of two or more numbers is the lowest multiple found in all the sets of multiples.
- A factor of a number divides into it exactly. The HCF of two or more numbers is the highest factor found in all the sets of factors.
- Prime numbers have only two factors, 1 and the number itself. The number 1 is not a prime number.
- A prime factor is a number that is both a factor and a prime number.
- All natural numbers that are not prime can be expressed as a product of prime factors.
- Integers are also called directed numbers. The sign of an integer (– or +) indicates whether its value is above or below 0.
- Mathematicians apply a standard set of rules to decide the order in which operations must be carried out. Operations in grouping symbols are worked out first, then division and multiplication, then addition and subtraction.

Are you able to . . . ?

- identify natural numbers, integers, square numbers and prime numbers
- find multiples and factors of numbers and identify the LCM and HCF
- write numbers as products of their prime factors using division and factor trees
- calculate squares, square roots, cubes and cube roots of numbers
- work with integers used in real-life situations
- apply the basic rules for operating with numbers
- perform basic calculations using mental methods and with a calculator.

Examination practice

Exam-style questions

1 Here is a set of numbers: {–4, –1, 0, 3, 4, 6, 9, 15, 16, 19, 20}
Which of these numbers are:

(a) natural numbers? **(b)** square numbers? **(c)** negative integers?
(d) prime numbers? **(e)** multiples of two? **(f)** factors of 80?

2 **(a)** List all the factors of 12. **(b)** List all the factors of 24. **(c)** Find the HCF of 12 and 24.

3 Find the HCF of 64 and 144.

4 List the first five multiples of:

(a) 12 **(b)** 18 **(c)** 30 **(d)** 80

5 Find the LCM of 24 and 36.

6 List all the prime numbers from 0 to 40.

7 **(a)** Use a factor tree to express 400 as a product of prime factors.
(b) Use the division method to express 1080 as a product of prime factors.
(c) Use your answers to find:

(i) the LCM of 400 and 1080 **(ii)** the HCF of 400 and 1080
(iii) $\sqrt{400}$ **(iv)** whether 1080 is a cube number; how can you tell?

8 Calculate:

(a) 26^2 **(b)** 43^3

9 What is the smallest number greater than 100 that is:

(a) divisible by two? **(b)** divisible by ten? **(c)** divisible by four?

10 At noon one day the outside temperature is 4 °C. By midnight the temperature is 8 degrees lower.
What temperature is it at midnight?

11 Simplify:

(a) $6 \times 2 + 4 \times 5$ **(b)** $4 \times (100 - 15)$ **(c)** $(5 + 6) \times 2 + (15 - 3 \times 2) - 6$

12 Add brackets to this statement to make it true.

$7 + 14 \div 4 - 1 \times 2 = 14$

Past paper questions

1 **(a)** Write down a common multiple of 8 and 14. [1]
(b) Complete the list of factors of 81: 1, …, …, …, 81 [2]
(c) Write down the prime factor of 81. [1]

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2 Calculate:

(a) $\dfrac{0.0548}{1.65 + 5.2 \times 7}$ **(b)** $\dfrac{0.0763}{1.85 + 4.7 \times 8}$ [2]

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2 Making sense of algebra

Key words

- Algebra
- Variable
- Equation
- Formula
- Substitution
- Expression
- Term
- Power
- Index
- Coefficient
- Exponent
- Base
- Reciprocal

In this chapter you will learn how to:

- use letters to represent numbers
- write expressions to represent mathematical information
- substitute letters with numbers to find the value of an expression
- add and subtract like terms to simplify expressions
- multiply and divide to simplify expressions
- expand expressions by removing grouping symbols
- use index notation in algebra
- learn and apply the laws of indices to simplify expressions.

EXTENDED
- work with fractional indices.

Once you know the basic rules, algebra is very easy and very useful.

You can think of **algebra** as the language of mathematics. Algebra uses letters and other symbols to write mathematical information in shorter ways.

When you learn a language, you have to learn the rules and structures of the language. The language of algebra also has rules and structures. Once you know these, you can 'speak' the language of algebra and mathematics students all over the world will understand you.

At school, and in the real world, you will use algebra in many ways. For example, you will use it to make sense of formulae and spreadsheets and you may use algebra to solve problems to do with money, building, science, agriculture, engineering, construction, economics and more.

2.1 Using letters to represent unknown values

In primary school you used empty shapes to represent unknown numbers. For example, 2 + ■ = 8 and ■ + ♦ = 10. If 2 + ■ = 8, the ■ can only represent 6. But if ■ + ♦ = 10, then the ■ and the ♦ can represent many different values.

In algebra the letters can represent many different values so they are called **variables**.

In algebra, you use letters to represent unknown numbers. So you could write the number sentences above as: $2 + x = 8$ and $a + b = 10$. Number sentences like these are called **equations**. You can solve an equation by finding the values that make the equation true.

If a problem introduces algebra, you must not change the 'case' of the letters used. For example, '*n*' and '*N*' can represent *different* numbers in the *same* formula!

When you worked with area of rectangles and triangles in the past, you used algebra to make a general rule, or **formula**, for working out the area, A:

Area of a rectangle = length × breadth, so $A = lb$

Area of a triangle = $\frac{1}{2}$base × height, so $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$

Notice that when two letters are multiplied together, we write them next to each other e.g lb, rather than $l \times b$.

To use a formula you have to replace some or all of the letters with numbers. This is called **substitution**.

Writing algebraic expressions

An algebraic **expression** is a group of letter and numbers linked by operation signs. Each part of the expression is called a **term**.

Suppose the average height (in centimetres) of students in your class is an unknown number, h. A student who is 10 cm taller than the average would have a height of $h + 10$. A student who is 3 cm shorter than the average would have a height of $h - 3$.

$h + 10$ and $h - 3$ are algebraic expressions. Because the unknown value is represented by h, we say these are expressions in terms of h.

Worked example 1

Use algebra to write an expression in terms of h for:

(a) a height 12 cm shorter than average
(b) a height $2x$ taller than average
(c) a height twice the average height
(d) a height half the average height.

(a)	$h - 12$	Shorter than means less than, so you subtract.
(b)	$h + 2x$	Taller than means more than, so you add. $2x$ is unknown, but it can still be used in the expression.
(c)	$2 \times h$	Twice means two times, so you multiply by two.
(d)	$h \div 2$	Half means divided by two.

Applying the rules

Algebraic expressions should be written in the shortest, simplest possible way.

Mathematicians write the product of a number and a variable with the number first to avoid confusion with powers. For example, $x \times 5$ is written as $5x$ rather than $x5$, which may be confused with x^5.

- $2 \times h$ is written as $2h$ and $x \times y$ is written as xy
- h means $1 \times h$, but you do not write the 1
- $h \div 2$ is written as $\frac{h}{2}$ and $x \div y$ is written as $\frac{x}{y}$
- when you have the product of a number and a variable, the number is written first, so $2h$ and not $h2$. Also, variables are normally written in alphabetical order, so xy and $2ab$ rather than yx and $2ba$

- $h \times h$ is written as h^2 (h squared) and $h \times h \times h$ is written as h^3 (h cubed). The ^{2} and the ^{3} are examples of a **power** or **index**.
- The power only applies to the number or variable directly before it, so $5a^2$ means $5 \times a \times a$
- When a power is outside a bracket, it applies to everything inside the bracket. So, $(xy)^3$ means $xy \times xy \times xy$

Worked example 2

Write expressions in terms of x to represent:

(a) a number times four
(b) the sum of the number and five
(c) six times the number minus two
(d) half the number.

(a)	x times 4 $= 4 \times x$ $= 4x$	Let x represent 'the number'. Replace 'four times' with $4 \times$. Leave out the $\times$ sign, write the number before the variable.
(b)	Sum of x and five $= x + 5$	Let x represent 'the number'. Sum of means +, replace five with 5.
(c)	Six times x minus two $= 6 \times x - 2$ $= 6x - 2$	Let x represent the number. Times means $\times$ and minus means $-$, insert numerals. Leave out the $\times$ sign.
(d)	Half x $= x \div 2$ $= \frac{x}{2}$	Let x represent 'the number'. Half means $\times \frac{1}{2}$ or $\div 2$. Write the division as a fraction.

Exercise 2.1

1 Rewrite each expression in its simplest form.

(a) $6 \times x \times y$ **(b)** $7 \times a \times b$ **(c)** $x \times y \times z$
(d) $2 \times y \times y$ **(e)** $a \times 4 \times b$ **(f)** $x \times y \times 12$
(g) $5 \times b \times a$ **(h)** $y \times z \times z$ **(i)** $6 \div x$
(j) $4x \div 2y$ **(k)** $(x + 3) \div 4$ **(l)** $m \times m \times m \div m \times m$
(m) $4 \times x + 5 \times y$ **(n)** $a \times 7 - 2 \times b$ **(o)** $2 \times x \times (x - 4)$
(p) $3 \times (x + 1) \div 2 \times x$ **(q)** $2 \times (x + 4) \div 3$ **(r)** $(4 \times x) \div (2 \times x + 4 \times x)$

REWIND
Remember BODMAS in Chapter 1. Work out the bit in brackets first. ◀

2 Let the unknown number be m. Write expressions for:

(a) the sum of the unknown number and 13
(b) a number that will exceed the unknown number by five
(c) the difference between 25 and the unknown number
(d) the unknown number cubed
(e) a third of the unknown number plus three
(f) four times the unknown number less twice the number.

REWIND
Remember from Chapter 1 that a 'sum' is the result of an addition. ◀

Also remember that the 'difference' between two numbers is the result of a subtraction. The order of the subtraction matters. ◀

3 Let the unknown number be x. Write expressions for:

(a) three more than x
(b) six less than x
(c) ten times x
(d) the sum of -8 and x
(e) the sum of the unknown number and its square

(f) a number which is twice x more than x
(g) the fraction obtained when double the unknown number is divided by the sum of the unknown number and four.

4 A CD and a DVD cost x dollars.

(a) If the CD costs \$10 what does the DVD cost?
(b) If the DVD costs three times the CD, what does the CD cost?
(c) If the CD costs $\$(x - 15)$, what does the DVD cost?

5 A woman is m years old.

(a) How old will she be in ten years' time?
(b) How old was she ten years ago?
(c) Her son is half her age. How old is the son?

6 Three people win a prize of $\$p$.

(a) If they share the prize equally, how much will each receive?
(b) If one of the people wins three times as much money as the other two, how much will each receive?

2.2 Substitution

When you substitute values you need to write in the operation signs. $5h$ means $5 \times h$, so if $h = 1$, or $h = 6$, you cannot write this in numbers as 51 or 56.

Expressions have different values depending on what numbers you substitute for the variables. For example, let's say casual waiters get paid \$5 per hour. You can write an expression to represent everyone's wages like this: $5h$, where h is the number of hours worked. If you work 1 hour, then you get paid $5 \times 1 = \$5$. So the expression $5h$ has a value of \$5 in this case. If you work 6 hours, you get paid $5 \times 6 = \$30$. The expression $5h$ has a value of \$30 in this case.

'Evaluate' means to find the value of.

Worked example 3

Given that $a = 2$ and $b = 8$, evaluate:

(a) ab **(b)** $3b - 2a$ **(c)** $2a^3$ **(d)** $2a(a + b)$

(a)	$ab = a \times b$ $= 2 \times 8$ $= 16$	Put back the multiplication sign. Substitute the values for a and b. Calculate the answer.
(b)	$3b - 2a = 3 \times b - 2 \times a$ $= 3 \times 8 - 2 \times 2$ $= 24 - 4$ $= 20$	Put back the multiplication signs. Substitute the values for a and b. Use the order of operations rules (× before –). Calculate the answer.
(c)	$2a^3 = 2 \times a^3$ $= 2 \times 2^3$ $= 2 \times 8$ $= 16$	Put back the multiplication signs. Substitute the value for a. Work out 2^3 first (grouping symbols first). Calculate the answer.
(d)	$2a(a + b) = 2 \times a \times (a + b)$ $= 2 \times 2 \times (2 + 8)$ $= 4 \times 10$ $= 40$	Put back the multiplication signs. Substitute the values for a and b. In this case you can carry out two steps at the same time: multiplication outside the bracket, and the addition inside. Calculate the answer.

REWIND

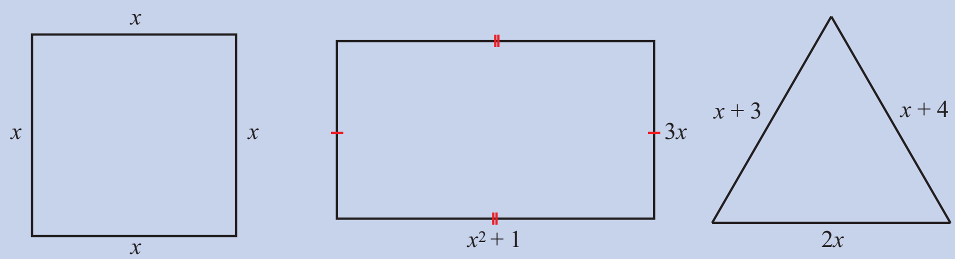
You will need to keep reminding yourself about the order (BODMAS) of operations from chapter 1. ◀

Worked example 4

For each of the shapes in the diagram below:

(i) Write an expression for the perimeter of each shape.
(ii) Find the perimeter in cm if $x = 4$.

(a) **(b)** **(c)**



(a)	**(i)**	$x + x + x + x = 4x$	Add the four lengths together.
	(ii)	$4 \times x = 4 \times 4$ $= 16\,\text{cm}$	Substitute 4 into the expression.
(b)	**(i)**	$3x + (x^2 + 1) + 3x + (x^2 + 1) = 2(3x) + 2(x^2 + 1)$	Add the four lengths together and write in its simplest form.
	(ii)	$2 \times (3 \times x) + 2 \times (x^2 + 1) = 2 \times (3 \times 4) + 2 \times (4^2 + 1)$ $= 2 \times 12 + 2 \times (16 + 1)$ $= 24 + 2 \times 17$ $= 24 + 34$ $= 58\,\text{cm}$	Substitute 4 into the expression.
(c)	**(i)**	$x + 3 + x + 4 + 2x$	Add the three lengths together.
	(ii)	$x + 3 + x + 4 + 2 \times x = 4 + 3 + 4 + 4 + 2 \times 4$ $= 4 + 3 + 4 + 4 + 8$ $= 23\,\text{cm}$	Substitute 4 into the expression.

Worked example 5

Complete this table of values for the formula $b = 3a - 3$

a	0	2	4	6
b				

a	0	2	4	6
b	–3	3	9	15

Substitute in the values of a to work out b.

$3 \times 0 - 3 = 0 - 3 = -3$

$3 \times 2 - 3 = 6 - 3 = 3$

$3 \times 4 - 3 = 12 - 3 = 9$

$3 \times 6 - 3 = 18 - 3 = 15$

Exercise 2.2

FAST FORWARD

You will learn more about algebraic fractions in chapter 14. ▶

1 Evaluate the following expressions for $x = 3$.

(a) $3x$ **(b)** $10x$ **(c)** $4x - 2$ **(d)** x^3 **(e)** $2x^2$

(f) $10 - x$ **(g)** $x^2 + 7$ **(h)** $x^3 + x^2$ **(i)** $2(x - 1)$ **(j)** $\frac{4x}{2}$

(k) $\frac{6x}{3}$ **(l)** $\frac{90}{x}$ **(m)** $\frac{10x}{6}$ **(n)** $\frac{(4x+2)}{7}$

Always show your substitution clearly. Write the formula or expression in its algebraic form but with the letters replaced by the appropriate numbers. This makes it clear to your teacher, or an examiner, that you have put the correct numbers in the right places.

2 What is the value of each expression when $a = 3$ and $b = 5$ and $c = 2$?

(a) abc	**(b)** a^2b	**(c)** $4a + 2c$	**(d)** $3b - 2(a + c)$
(e) $a^2 + c^2$	**(f)** $4b - 2a + c$	**(g)** $ab + bc + ac$	**(h)** $2(ab)^2$
(i) $3(a + b)$	**(j)** $(b - c) + (a + c)$	**(k)** $(a + b)(b - c)$	**(l)** $\frac{3bc}{ac}$
(m) $\frac{4b}{a} + c$	**(n)** $\frac{4b^2}{bc}$	**(o)** $\frac{2(a+b)}{c^2}$	**(p)** $\frac{3abc}{10a}$
(q) $\frac{6b^2}{(a+c)^2}$	**(r)** $(\frac{1}{2}abc)^2$	**(s)** $\frac{8a}{\sqrt[3]{a+b}}$	**(t)** $\frac{6ab}{a^2} - 2bc$

You may need to discuss part (f)(i) with your teacher.

3 Work out the value of y in each formula when:

(i) $x = 0$ **(ii)** $x = 3$ **(iii)** $x = 4$ **(iv)** $x = 10$ **(v)** $x = 50$

(a) $y = 4x$	**(b)** $y = 3x + 1$	**(c)** $y = 100 - x$
(d) $y = \frac{x}{2}$	**(e)** $y = x^2$	**(f)** $y = \frac{100}{x}$
(g) $y = 2(x + 2)$	**(h)** $y = 2(x + 2) - 10$	**(i)** $y = 3x^3$

4 A sandwich costs \$3 and a drink costs \$2.

(a) Write an expression to show the total cost of buying x sandwiches and y drinks.
(b) Find the total cost of:

(i) four sandwiches and three drinks
(ii) 20 sandwiches and 20 drinks
(iii) 100 sandwiches and 25 drinks.

5 The formula for finding the perimeter of a rectangle is $P = 2(l + b)$, where l represents the length and b represents the breadth of the rectangle.

Find the perimeter of a rectangle if:

(a) the length is 12 cm and the breadth is 9 cm
(b) the length is 2.5 m and the breadth is 1.5 m
(c) the length is 20 cm and the breadth is half as long
(d) the breadth is 2 cm and the length is the cube of the breadth.

2.3 Simplifying expressions

Remember, terms are not separated by × or ÷ signs. A fraction line means divide, so the parts of a fraction are all counted as one term, even if there is a + or – sign in the numerator or denominator.

So, $\frac{a+b}{c}$ is one term.

Remember, the number in a term is called a **coefficient**. In the term $2a$, the coefficient is 2; in the term $-3ab$, the coefficient is –3. A term with only numbers is called a constant. So in $2a + 4$, the constant is 4.

The parts of an algebraic expression are called *terms*. Terms are separated from each other by + or – signs. So $a + b$ is an expression with two terms, but ab is an expression with only one term and $2 + \frac{3a}{b} - \frac{ab}{c}$ is an expression with three terms.

Adding and subtracting like terms

Terms with exactly the same variables are called *like terms*. $2a$ and $4a$ are like terms; $3xy^2$ and $-xy^2$ are like terms.

The variables and any indices attached to them have to be identical for terms to be like terms. Don't forget that variables in a different order mean the same thing, so xy and yx are like terms ($x \times y = y \times x$).

Like terms can be added or subtracted to simplify algebraic expressions.

Note that a '+' or a '–' that appears within an algebraic expression, is attached to the term that sits to its right. For example: $3x - 4y$ contains two terms, $3x$ and $-4y$. If a term has no symbol written before it then it is taken to mean that it is '+'.

Worked example 6

Simplify:

(a) $4a + 2a + 3a$ (b) $4a + 6b + 3a$ (c) $5x + 2y - 7x$
(d) $2p + 5q + 3q - 7p$ (e) $2ab + 3a^2b - ab + 3ab^2$

(a)	$4a + 2a + 3a$ $= 9a$	Terms are all like. Add the coefficients, write the term.
(b)	$4a + 6b + 3a$ $= 7a + 6b$	Identify the like terms ($4a$ and $3a$). Add the coefficients of like terms. Write terms in alphabetical order.
(c)	$5x + 2y - 7x$ $= -2x + 2y$	Identify the like terms ($5x$ and $-7x$). Subtract the coefficients, remember the rules. Write the terms. (This could also be written as $2y - 2x$.)
(d)	$2p + 5q + 3q - 7p$ $= -5p + 8q$	Identify the like terms ($2p$ and $-7p$; $5q$ and $3q$). Add and subtract the coefficients. Write the terms.
(e)	$2ab + 3a^2b - ab + 3ab^2$ $= ab + 3a^2b + 3ab^2$	Identify like terms; pay attention to terms that are squared because a and a^2 are not like terms. Remember that ab means $1ab$.

Notice that you can rearrange the terms provided that you remember to take the '–' and '+' signs with the terms to their right. For example:

$3x - 2y + 5z$
$= 3x + 5z - 2y$
$= 5z + 3x - 2y$
$= -2y + 3x + 5z$

Exercise 2.3

1 Identify the like terms in each set.

(a) $6x, -2y, 4x, x$ (b) $x, -3y, \frac{3}{4}y, -5y$ (c) $ab, 4b, -4ba, 6a$
(d) $2, -2x, 3xy, 3x, -2y$ (e) $5a, 5ab, ab, 6a, 5$ (f) $-1xy, -yx, -2y, 3, 3x$

2 Simplify by adding or subtracting like terms.

(a) $2y + 6y$ (b) $9x - 2x$ (c) $10x + 3x$
(d) $21x + x$ (e) $7x - 2x$ (f) $4y - 4y$
(g) $9x - 10x$ (h) $y - 4y$ (i) $5x - x$
(j) $9xy - 2xy$ (k) $6pq - 2qp$ (l) $14xyz - xyz$
(m) $4x^2 - 2x^2$ (n) $9y^2 - 4y^2$ (o) $y^2 - 2y^2$
(p) $14ab^2 - 2ab^2$ (q) $9x^2y - 4x^2y$ (r) $10xy^2 - 8xy^2$

FAST FORWARD

You will need to be very comfortable with the simplification of algebraic expressions when solving equations, inequalities and simplifying expansions throughout the course. ▶

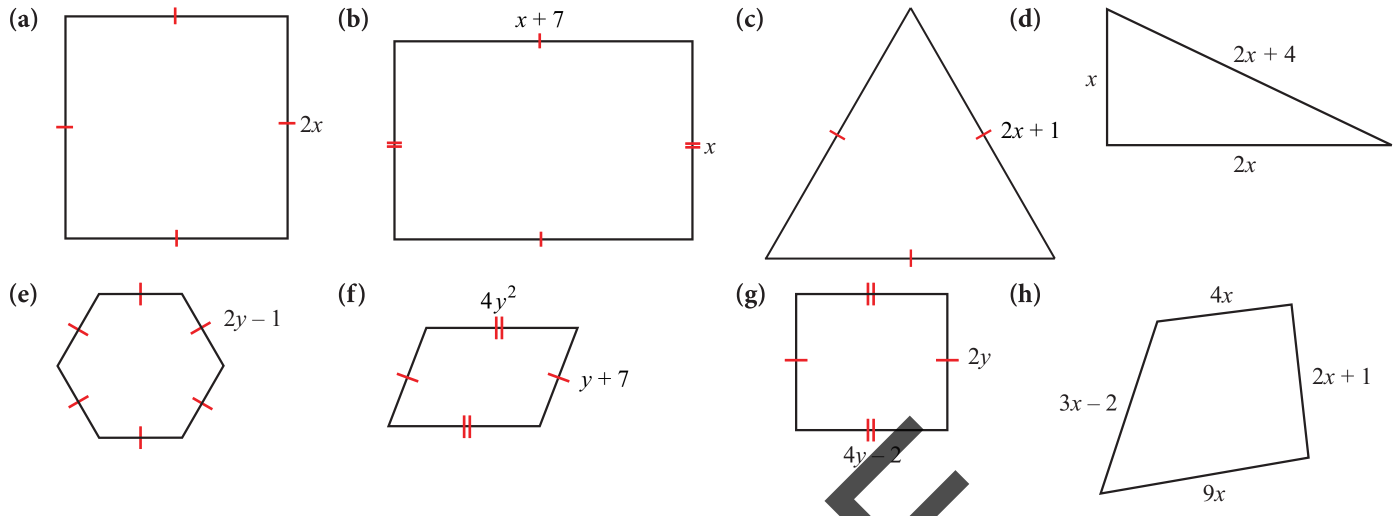
3 Simplify:

(a) $2x + y + 3x$ (b) $4y - 2y + 4x$ (c) $6x - 4x + 5x$
(d) $10 + 4x - 6$ (e) $4xy - 2y + 2xy$ (f) $5x^2 - 6x^2 + 2x$
(g) $5x + 4y - 6x$ (h) $3y + 4x - x$ (i) $4x + 6y + 4x$
(j) $9x - 2y - x$ (k) $12x^2 - 4x + 2x^2$ (l) $12x^2 - 4x^2 + 2x^2$
(m) $5xy - 2x + 7xy$ (n) $xy - 2xz + 7xy$ (o) $3x^2 - 2y^2 - 4x^2$
(p) $5x^2y + 3x^2y - 2xy$ (q) $4xy - x + 2yx$ (r) $5xy - 2 + xy$

4 Simplify as far as possible:

(a) $8y - 4 - 6y - 4$ (b) $x^2 - 4x + 3x^2 - x$ (c) $5x + y + 2x + 3y$
(d) $y^2 + 2y + 3y - 7$ (e) $x^2 - 4x - x + 3$ (f) $x^2 + 3x - 7 + 2x$
(g) $4xyz - 3xy + 2xz - xyz$ (h) $5xy - 4 + 3yx - 6$ (i) $8x - 4 - 2x - 3x^2$

5 Write an expression for the perimeter (P) of each of the following shapes and then simplify it to give P in the simplest possible terms.



Multiplying and dividing in expressions

Although terms are not separated by × or ÷ they still need to be written in the simplest possible way to make them easier to work with.

In section 2.1 you learned how to write expressions in simpler terms when multiplying and dividing them. Make sure you understand and remember these important rules:

- $3x$ means $3 \times x$ and $3xy$ means $3 \times x \times y$
- xy means $x \times y$
- x^2 means $x \times x$ and x^2y means $x \times x \times y$ (only the x is squared)
- $\frac{2a}{4}$ means $2a \div 4$

Worked example 7

Simplify:

(a) $4 \times 3x$ **(b)** $4x \times 3y$ **(c)** $4ab \times 2bc$ **(d)** $7x \times 4yz \times 3$

(a)	$4 \times 3x = 4 \times 3 \times x$ $= 12 \times x$ $= 12x$	Insert the missing × signs. Multiply the numbers first. Write in simplest form.
(b)	$4x \times 3y = 4 \times x \times 3 \times y$ $= 12 \times x \times y$ $= 12xy$	Insert the missing × signs. Multiply the numbers. Write in simplest form.
(c)	$4ab \times 2bc = 4 \times a \times b \times 2 \times b \times c$ $= 8 \times a \times b \times b \times c$ $= 8ab^2c$	Insert the missing × signs. Multiply the numbers, then the variables. Write in simplest form.
(d)	$7x \times 4yz \times 3 = 7 \times x \times 4 \times y \times z \times 3$ $= 84 \times x \times y \times z$ $= 84xyz$	Insert the missing × signs. Multiply the numbers. Write in simplest form.

You can multiply numbers first and variables second because the order of any multiplication can be reversed without changing the answer.

FAST FORWARD

You will learn more about cancelling and equivalent fractions in chapter 5. ▶

Worked example 8

Simplify:

(a) $\frac{12x}{3}$ (b) $\frac{12xy}{3x}$ (c) $\frac{7xy}{70y}$ (d) $\frac{2x}{3}\times\frac{4x}{2}$

(a)	$\frac{12x}{3}=\frac{\overset{4}{\cancel{12}}x}{\underset{1}{\cancel{3}}}=\frac{4x}{1}=4x$	Divide both top and bottom by 3 (making the numerator and denominator smaller so that the fraction is in its simplest form is called *cancelling*).
(b)	$\frac{12xy}{3x}=\frac{\overset{4}{\cancel{12}}\cancel{x}y}{\underset{1}{\cancel{3}}\cancel{x}}=\frac{4\times y}{1}=4y$	Cancel and then multiply.
(c)	$\frac{7xy}{70y}=\frac{\overset{1}{\cancel{7}}x\cancel{y}}{\underset{10}{\cancel{70}}\cancel{y}}=\frac{x}{10}$	Cancel.
(d)	$\frac{2x}{3}\times\frac{4x}{2}=\frac{2\times x\times 4\times x}{3\times 2}$	Insert signs and multiply.
	$=\frac{\overset{4}{\cancel{8}}x^2}{\cancel{6}_3}$	Cancel.
	$=\frac{4x^2}{3}$	
	or	
	$\frac{\overset{1}{\cancel{2}}x}{3}\times\frac{4x}{\underset{1}{\cancel{2}}}=\frac{1x}{3}\times\frac{4x}{1}=\frac{4x^2}{3}$	Cancel first, then multiply.

Exercise 2.4

1 Multiply:

(a) $2\times 6x$ (b) $4y\times 2$ (c) $3m\times 4$
(d) $2x\times 3y$ (e) $4x\times 2y$ (f) $9x\times 3y$
(g) $8y\times 3z$ (h) $2x\times 3y\times 2$ (i) $4xy\times 2xy$
(j) $4xy\times 2x$ (k) $9y\times 3xy$ (l) $4y\times 2x\times 3y$
(m) $2a\times 4ab$ (n) $3ab\times 4bc$ (o) $6abc\times 2a$
(p) $8abc\times 2ab$ (q) $4\times 2ab\times 3c$ (r) $12x^2\times 2\times 3y^2$

2 Simplify:

(a) $3\times 2x\times 4$ (b) $5x\times 2x\times 3y$ (c) $2x\times 3y\times 2xy$
(d) $xy\times xz\times x$ (e) $2\times 2\times 3x\times 4$ (f) $4\times 2x\times 3x^2y$
(g) $x\times y^2\times 4x$ (h) $2a\times 3ab\times 2c$ (i) $10x\times 2y\times 3$
(j) $4\times x\times 2\times y$ (k) $9\times x^2\times xy$ (l) $4xy^2\times 2x^2y$
(m) $7xy\times 2xz\times 3yz$ (n) $4xy\times 2x^2y\times 7$ (o) $9\times xyz\times 4xy$
(p) $3x^2y\times 2xy^2\times 3xy$ (q) $9x\times 2xy\times 3x^2$ (r) $2x\times xy^2\times 3xy$

3 Simplify:

(a) $\frac{15x}{3}$ (b) $\frac{40x}{10}$ (c) $\frac{21x}{7}$ (d) $\frac{12xy}{2x}$
(e) $\frac{14xy}{2y}$ (f) $\frac{18x^2y}{9x^2}$ (g) $\frac{10xy}{40x}$ (h) $\frac{15x}{60xy}$
(i) $\frac{7xyz}{14xy}$ (j) $\frac{6xy}{x}$ (k) $\frac{x}{4x}$ (l) $\frac{x}{9x}$

4 Simplify:

(a) $8x \div 2$ (b) $12xy \div 2x$ (c) $16x^2 \div 4xy$ (d) $24xy \div 3xy$

(e) $14x^2 \div 2y^2$ (f) $24xy \div 8y$ (g) $8xy \div 24y$ (h) $9x \div 36xy$

(i) $\frac{77xyz}{11xz}$ (j) $\frac{45xy}{20x}$ (k) $\frac{60x^2y^2}{15xy}$ (l) $\frac{100xy}{25x^2}$

5 Simplify these as far as possible.

(a) $\frac{x}{2} \times \frac{y}{3}$ (b) $\frac{x}{3} \times \frac{x}{4}$ (c) $\frac{xy}{2} \times \frac{5x}{3}$ (d) $\frac{2x}{3} \times \frac{5}{y}$

(e) $\frac{2x}{4} \times \frac{3y}{4}$ (f) $\frac{5x}{2} \times \frac{5x}{2}$ (g) $\frac{x}{y} \times \frac{2y}{x}$ (h) $\frac{xy}{3} \times \frac{x}{y}$

(i) $5y \times \frac{2x}{5}$ (j) $4 \times \frac{2x}{3}$ (k) $\frac{x}{6} \times \frac{3}{2x}$ (l) $\frac{5x}{2} \times \frac{4x}{10}$

2.4 Working with brackets

FAST FORWARD

In this section you will focus on simple examples. You will learn more about removing brackets and working with negative terms in chapters 6 and 10. You will also learn a little more about why this method works. ▶

Removing brackets is really just multiplying, so the same rules you used for multiplication apply in these examples.

When an expression has brackets, you normally have to remove the brackets before you can simplify the expression. Removing the brackets is called expanding the expression.

To remove brackets you multiply each term inside the bracket by the number (and/or variables) outside the bracket. When you do this you need to pay attention to the positive and negative signs in front of the terms:

$x(y + z) = xy + xz$

$x(y - z) = xy - xz$

Worked example 9

Remove the brackets to simplify the following expressions.

(a) $2(2x + 6)$ (b) $4(7 - 2x)$ (c) $2x(x + 3y)$ (d) $xy(2 - 3x)$

(a) $2(2x + 6) = 2 \times 2x + 2 \times 6$
$= 4x + 12$

(b) $4(7 - 2x) = 4 \times 7 - 4 \times 2x$
$= 28 - 8x$

(c) $2x(x + 3y) = 2x \times x + 2x \times 3y$
$= 2x^2 + 6xy$

(d) $xy(2 - 3x) = xy \times 2 - xy \times 3x$
$= 2xy - 3x^2y$

For parts (a) to (d) write the expression out, or do the multiplication mentally.

Follow these steps when multiplying by a term outside a bracket:

- Multiply the term on the left-hand inside of the bracket first - shown by the red arrow labelled i.
- Then multiply the term on the right-hand side – shown by the blue arrow labelled ii.
- Then add the answers together.

Exercise 2.5

1 Expand:

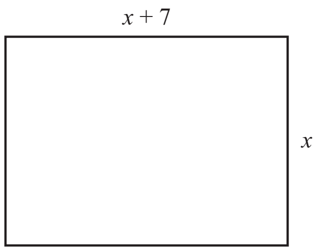
(a) $2(x+6)$ **(b)** $3(x+2)$ **(c)** $4(2x+3)$
(d) $10(x-6)$ **(e)** $4(x-2)$ **(f)** $3(2x-3)$
(g) $5(y+4)$ **(h)** $6(4+y)$ **(i)** $9(y+2)$
(j) $7(2x-2y)$ **(k)** $2(3x-2y)$ **(l)** $4(x+4y)$
(m) $5(2x-2y)$ **(n)** $6(3x-2y)$ **(o)** $3(4y-2x)$
(p) $4(y-4x^2)$ **(q)** $9(x^2-y)$ **(r)** $7(4x+x^2)$

2 Remove the brackets to expand these expressions.

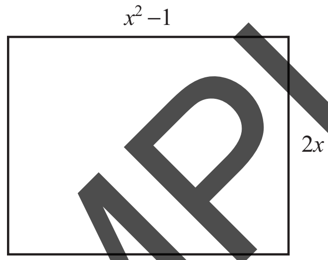
(a) $2x(x+y)$ **(b)** $3y(x-y)$ **(c)** $2x(x+2y)$
(d) $4x(3x-2y)$ **(e)** $xy(x-y)$ **(f)** $3y(4x+2)$
(g) $2xy(9-4y)$ **(h)** $2x^2(3-2y)$ **(i)** $3x^2(4-4x)$
(j) $4x(9-2y)$ **(k)** $5y(2-x)$ **(l)** $3x(4-y)$
(m) $2x^2y(y-2x)$ **(n)** $4xy^2(3-2x)$ **(o)** $3xy^2(x+y)$
(p) $x^2y(2x+y)$ **(q)** $9x^2(9-2x)$ **(r)** $4xy^2(3-x)$

3 Given the formula for area, $A = \text{length} \times \text{breadth}$, write an expression for A in terms of x for each of the following rectangles. Expand the expression to give A in simplest terms.

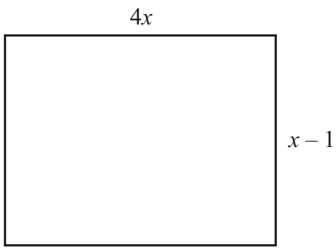
(a)



(b)



(c)



Expanding and collecting like terms

When you remove brackets and expand an expression you may end up with some like terms. When this happens, you collect the like terms together and add or subtract them to write the expression in its simplest terms.

Worked example 10

Expand and simplify where possible.

(a) $6(x+3)+4$ **(b)** $2(6x+1)-2x+4$ **(c)** $2x(x+3)+x(x-4)$

(a)	$6(x+3)+4 = 6x+18+4$ $= 6x+22$	Remove the brackets. Add like terms.
(b)	$2(6x+1)-2x+4 = 12x+2-2x+4$ $= 10x+6$	Remove the brackets. Add or subtract like terms.
(c)	$2x(x+3)+x(x-4) = 2x^2+6x+x^2-4x$ $= 3x^2+2x$	Remove the brackets. Add or subtract like terms.

Exercise 2.6

1 Expand and simplify:

(a) $2(5+x)+3x$ **(b)** $3(y-2)+4y$ **(c)** $2x+2(x-4)$
(d) $4x+2(x-3)$ **(e)** $2x(4+x)-5$ **(f)** $4(x+2)-7$

(g) $6+3(x-2)$ **(h)** $4x+2(2x+3)$ **(i)** $2x+3+2(2x+3)$
(j) $3(2x+2)-3x-4$ **(k)** $6x+2(x+3)$ **(l)** $7y+y(x-4)-4$
(m) $2x(x+4)-4$ **(n)** $2y(2x-2y+4)$ **(o)** $2y(5-4y)-4y^2$
(p) $3x(2x+4)-9$ **(q)** $3y(y+2)-4y^2$ **(r)** $2(x-1)+4x-4$

2 Simplify these expressions by removing brackets and collecting like terms.

(a) $4(x+40)+2(x-3)$ **(b)** $2(x-2)+2(x+3)$ **(c)** $3(x+2)+4(x+5)$
(d) $8(x+10)+4(3-2x)$ **(e)** $4(x^2+2)+2(4-x^2)$ **(f)** $4x(x+1)+2x(x+3)$
(g) $3x(4y-4)+4(3xy+4x)$ **(h)** $2x(5y-4)+2(6x-4xy)$ **(i)** $3x(4-8y)+3(2xy-5x)$
(j) $3(6x-4y)+x(3-2y)$ **(k)** $3x^2(4-x)+2(5x^2-2x^3)$ **(l)** $x(x-y)+3(2x-y)$
(m) $4(x-2)+3x(4-y)$ **(n)** $x(x+y)+x(x-y)$ **(o)** $2x(x+y)+2(x^2+3xy)$
(p) $x(2x+3)+3(5-2x)$ **(q)** $4(2x-3)+(x-5)$ **(r)** $3(4xy-2x)+5(3x-xy)$

2.5 Indices

Revisiting index notation

The plural of 'index' is 'indices'.

You already know how to write powers of two and three using indices:

$2\times2=2^2$ and $y\times y=y^2$

$2\times2\times2=2^3$ and $y\times y\times y=y^3$

Exponent is another word sometimes used to mean 'index' or 'power'. These words can be used interchangeably but 'index' is more commonly used for IGCSE.

When you write a number using indices (powers) you have written it in index notation. Any number can be used as an index including 0, negative integers and fractions. The index tells you how many times the **base** has been multiplied by itself. So:

$3\times3\times3\times3=3^4$ 3 is the base, 4 is the index

$a\times a\times a\times a\times a=a^5$ a is the base, 5 is the index

Worked example 11

Write each expression using index notation.

(a) $2\times2\times2\times2\times2\times2$ **(b)** $x\times x\times x\times x$ **(c)** $x\times x\times x\times y\times y\times y\times y$

When you write a power out in full as a multiplication you are writing it in expanded form.

(a)	$2\times2\times2\times2\times2\times2=2^6$	Count how many times 2 is multiplied by itself to give you the index.
(b)	$x\times x\times x\times x=x^4$	Count how many times x is mulitplied by itself to give you the index.
(c)	$x\times x\times x\times y\times y\times y\times y=x^3y^4$	Count how many times x is mulitplied by itself to get the index of x; then work out the index of y in the same way.

Worked example 12

Use your calculator to evaluate:

(a) 2^5 **(b)** 2^8 **(c)** 10^6 **(d)** 7^4

When you *evaluate* a number raised to a power, you are carrying out the multiplication to obtain a single value.

(a)	$2^5=32$	Enter [2] [$x^{\square}$] [5] [=]
(b)	$2^8=256$	Enter [2] [$x^{\square}$] [8] [=]
(c)	$10^6=1\,000\,000$	Enter [1] [0] [$x^{\square}$] [6] [=]
(d)	$7^4=2401$	Enter [7] [$x^{\square}$] [4] [=]

REWIND

Quickly remind yourself, from chapter 1, how a composite number can be written as a product of primes. ◀

Index notation and products of prime factors

Index notation is very useful when you have to express a number as a product of its prime factors because it allows you to write the factors in a short form.

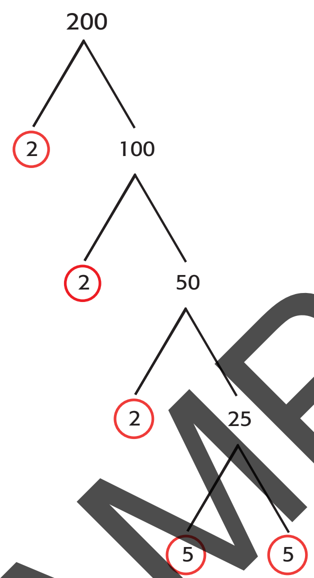
Worked example 13

Express these numbers as products of their prime factors in index form.

(a) 200 **(b)** 19 683

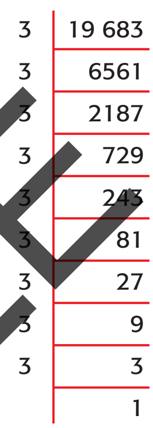
The diagrams below are a reminder of the factor tree and division methods for finding the prime factors.

(a)



$= 2 \times 2 \times 2 \times 5 \times 5$

(b)



$= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$

(a) $200 = 2^3 \times 5^2$ **(b)** $19683 = 3^9$

Exercise 2.7

1 Write each expression using index notation.

(a) $2\times2\times2\times2\times2$ **(b)** $3\times3\times3\times3$ **(c)** 7×7
(d) $11\times11\times11$ **(e)** $10\times10\times10\times10\times10$ **(f)** $8\times8\times8\times8\times8$
(g) $a\times a\times a\times a$ **(h)** $x\times x\times x\times x\times x$ **(i)** $y\times y\times y\times y\times y\times y$
(j) $a\times a\times a\times b\times b$ **(k)** $x\times x\times y\times y\times y\times y$ **(l)** $p\times p\times p\times q\times q$
(m) $x\times x\times x\times x\times x\times y\times y\times y$ **(n)** $x\times y\times x\times y\times y\times x\times y$ **(o)** $a\times b\times a\times b\times a\times b\times c$

2 Evaluate:

(a) 10^4 **(b)** 7^3 **(c)** 6^7 **(d)** 4^9 **(e)** 10^5
(f) 1^{12} **(g)** 2^{10} **(h)** 9^4 **(i)** 2^6 **(j)** $2^3\times3^4$
(k) $5^2\times3^8$ **(l)** $4^5\times2^6$ **(m)** $2^6\times3^4$ **(n)** $2^8\times3^2$ **(o)** $5^3\times3^5$

3 Express the following as products of prime factors, in index notation.

(a) 64 **(b)** 243 **(c)** 400 **(d)** 1600 **(e)** 16 384
(f) 20 736 **(g)** 59 049 **(h)** 390 625

4 Write several square numbers as products of prime factors, using index notation. What is true about the index needed for each prime?

The laws of indices

The laws of indices are very important in algebra because they give you quick ways of simplifying expressions. You will use these laws over and over again as you learn more and more algebra, so it is important that you understand them and that you can apply them in different situations.

Multiplying the same base number with different indices

Look at these two multiplications:

$3^2 \times 3^4 \qquad x^3 \times x^4$

In the first multiplication, 3 is the 'base' number and in the second, x is the 'base' number.

You already know you can simplify these by expanding them like this:

$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 \qquad x \times x \times x \times x \times x \times x \times x = x^7$

In other words:

$3^2 \times 3^4 = 3^{2+4}$ and $x^3 \times x^4 = x^{3+4}$

This gives you the law of indices for multiplication:

> When you multiply index expressions with the same base you can add the indices: $x^m \times x^n = x^{m+n}$

Worked example 14

Simplify:

(a) $4^3 \times 4^6$ **(b)** $x^2 \times x^3$ **(c)** $2x^2y \times 3xy^4$

(a)	$4^3 \times 4^6 = 4^{3+6} = 4^9$	Add the indices.
(b)	$x^2 \times x^3 = x^{2+3} = x^5$	Add the indices.
(c)	$2x^2y \times 3xy^4 = 2 \times 3 \times x^{2+1} \times y^{1+4} = 6x^3y^5$	Multiply the numbers first, then add the indices of like variables.

Remember every letter or number has a power of 1 (usually unwritten). So x means x^1 and y means y^1.

Dividing the same base number with different indices

FAST FORWARD

The multiplication and division rules will be used more when you study standard form in chapter 5. ▶

Look at these two divisions:

$3^4 \div 3^2$ and $x^6 \div x^2$

You already know you can simplify these by writing them in expanded form and cancelling like this:

$\frac{3 \times 3 \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3}} \qquad \frac{x \times x \times x \times x \times \cancel{x} \times \cancel{x}}{\cancel{x} \times \cancel{x}}$

$= 3 \times 3 \qquad = x \times x \times x \times x$

$= 3^2 \qquad = x^4$

In other words:

$3^4 \div 3^2 = 3^{4-2}$ and $x^6 \div x^2 = x^{6-2}$

This gives you the law of indices for division:

> When you divide index expressions with the same base you can subtract the indices: $x^m \div x^n = x^{m-n}$

Worked example 15

Simplify:

(a) $\frac{x^6}{x^2}$ (b) $\frac{6x^5}{3x^2}$ (c) $\frac{10x^3y^2}{5xy}$

(a)	$\frac{x^6}{x^2} = x^{6-2} = x^4$	Subtract the indices.
(b)	$\frac{6x^5}{3x^2} = \frac{6}{3} \times \frac{x^5}{x^2} = \frac{2}{1} \times x^{5-2} = 2x^3$	Divide (cancel) the coefficients. Subtract the indices.
(c)	$\frac{10x^3y^2}{5xy} = \frac{10}{5} \times \frac{x^3}{x} \times \frac{y^2}{y}$ $= \frac{2}{1} \times x^{3-1} \times y^{2-1}$ $= 2x^2y$	Divide the coefficients. Subtract the indices.

Remember 'coefficient' is the *number* in the term.

The power 0

You should remember that any value divided by itself gives 1.

So, $3 \div 3 = 1$ and $x \div x = 1$ and $\frac{x^4}{x^4} = 1$.

If we use the law of indices for division we can see that:

$$\frac{x^4}{x^4} = x^{4-4} = x^0$$

This gives us the law of indices for the power 0.

Technically, there is an awkward exception to this rule when $x = 0$. 0^0 is usually defined to be 1!

> Any value to the power 0 is equal to 1. So $x^0 = 1$.

Raising a power

Look at these two examples:

$(x^3)^2 = x^3 \times x^3 = x^{3+3} = x^6$ $(2x^3)^4 = 2x^3 \times 2x^3 \times 2x^3 \times 2x^3 = 2^4 \times x^{3+3+3+3} = 16x^{12}$

If we write the examples in expanded form we can see that $(x^3)^2 = x^6$ and $(2x^3)^4 = 16x^{12}$

This gives us the law of indices for raising a power to another power:

> When you have to raise a power to another power you multiply the indices: $(x^m)^n = x^{mn}$

Worked example 16

Simplify:

(a) $(x^3)^6$ (b) $(3x^4y^3)^2$ (c) $(x^3)^4 \div (x^6)^2$

(a)	$(x^3)^6 = x^{3\times6}$ $= x^{18}$	Multiply the indices.

A common error is to forget to take powers of the numerical terms. For example in part (b), the '3' needs to be squared to give '9'.

(b)	$(3x^4y^3)^2$ $= 3^2 \times x^{4\times 2} \times y^{3\times 2}$ $= 9x^8y^6$	Square each of the terms to remove the brackets and multiply the indices.
(c)	$(x^3)^4 \div (x^6)^2$ $= x^{3\times 4} \div x^{6\times 2}$ $= x^{12} \div x^{12}$ $= x^{12-12}$ $= x^0$ $= 1$	Expand the brackets first by multiplying the indices. Divide by subtracting the indices.

Exercise 2.8

1 Simplify:

(a) $3^2 \times 3^6$ **(b)** $4^2 \times 4^9$ **(c)** $8^2 \times 8^0$ **(d)** $x^9 \times x^4$
(e) $y^2 \times y^7$ **(f)** $y^3 \times y^4$ **(g)** $y \times y^5$ **(h)** $x \times x^4$
(i) $3x^4 \times 2x^3$ **(j)** $3y^2 \times 3y^4$ **(k)** $2x \times x^3$ **(l)** $3x^3 \times 2x^4$
(m) $5x^3 \times 3$ **(n)** $8x^4 \times x^3$ **(o)** $4x^6 \times 2x$ **(p)** $x^3 \times 4x^5$

2 Simplify:

(a) $x^6 \div x^4$ **(b)** $x^{12} \div x^3$ **(c)** $y^4 \div y^3$ **(d)** $x^3 \div x$ **(e)** $\frac{x^5}{x}$
(f) $\frac{x^6}{x^4}$ **(g)** $\frac{6x^5}{2x^3}$ **(h)** $\frac{9x^7}{3x^4}$ **(i)** $\frac{12y^2}{3y}$ **(j)** $\frac{3x^4}{6x^3}$
(k) $\frac{15x^3}{5x^3}$ **(l)** $\frac{9x^4}{3x^3}$ **(m)** $\frac{3x^3}{9x^4}$ **(n)** $\frac{16x^2y^2}{4xy}$ **(o)** $\frac{12xy^2}{12xy^2}$

3 Simplify:

(a) $(x^2)^2$ **(b)** $(x^2)^3$ **(c)** $(x^2)^6$ **(d)** $(y^3)^2$ **(e)** $(2x^2)^5$
(f) $(3x^2y^2)^2$ **(g)** $(x^4)^0$ **(h)** $(5x^2)^3$ **(i)** $(x^2y^2)^3$ **(j)** $(x^2y^4)^5$
(k) $(xy^4)^3$ **(l)** $(4xy^2)^2$ **(m)** $(3x^2)^4$ **(n)** $(xy^6)^4$ **(o)** $\left(\frac{x^2}{y}\right)^0$

4 Use the appropriate laws of indices to simplify these expressions.

When there is a mixture of numbers and letters, deal with the numbers first and then apply the laws of indices to the letters in alphabetical order.

(a) $2x^2 \times 3x^3 \times 2x$ **(b)** $4 \times 2x \times 3x^2y$ **(c)** $4x \times x \times x^2$
(d) $(x^2)^2 \div 4x^2$ **(e)** $11x^3 \times 4(a^2b)^2$ **(f)** $4x(x^2+7)$
(g) $x^2(4x - x^3)$ **(h)** $x^8 \div (x^3)^2$ **(i)** $7x^2y^2 \div (x^3y)^2$
(j) $\frac{(4x^2 \times 3x^4)}{6x^4}$ **(k)** $\left(\frac{x^4}{y^2}\right)^3$ **(l)** $\frac{x^8 \times (xy^2)^4}{(2x^2)^4}$
(m) $(8x^2)^0$ **(n)** $4x^2 \times 2x^3 \div (2x)^0$ **(o)** $\frac{(4x^2y^3)^2}{(2xy)^3}$

Negative indices

At the beginning of this unit you read that negative numbers can also be used as indices. But what does it mean if an index is negative?

Look at the two methods of working out $x^3 \div x^5$ below.

Using expanded notation:

$$x^3 \div x^5 = \frac{x \times x \times x}{x \times x \times x \times x \times x} = \frac{1}{x \times x} = \frac{1}{x^2}$$

Using the law of indices for division:

$$x^3 \div x^5 = x^{3-5} = x^{-2}$$

In plain language you can say that when a number is written with a negative power, it is equal to 1 over the number to the same positive power. Another way of saying '1 over' is **reciprocal**, so a^{-2} can be written as the reciprocal of a^2, i.e. $\frac{1}{a^2}$.

This shows that $\frac{1}{x^2} = x^{-2}$. And this gives you a rule for working with negative indices:

$$x^{-m} = \frac{1}{x^m} \quad (\text{when } x \neq 0)$$

When an expression contains negative indices you apply the same laws as for other indices to simplify it.

FAST FORWARD

These are simple examples. Once you have learned more about working with directed numbers in algebra in chapter 6, you will apply what you have learned to simplify more complicated expressions. ▶

Worked example 17

1 Find the value of:

(a) 4^{-2} **(b)** 5^{-1}

(a) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$ **(b)** $5^{-1} = \frac{1}{5^1} = \frac{1}{5}$

2 Write these with a positive index.

(a) x^{-4} **(b)** y^{-3}

(a) $x^{-4} = \frac{1}{x^4}$ **(b)** $y^{-3} = \frac{1}{y^3}$

3 Simplify. Give your answers with positive indices.

(a) $\frac{4x^2}{2x^4}$ **(b)** $2x^{-2} \times 3x^{-4}$ **(c)** $(3y^2)^{-3}$

(a) $\frac{4x^2}{2x^4} = \frac{4}{2} \times x^{2-4} = 2x^{-2} = \frac{2}{x^2}$

(b) $2x^{-2} \times 3x^{-4} = \frac{2}{x^2} \times \frac{3}{x^4} = \frac{6}{x^{2+4}} = \frac{6}{x^6}$

(c) $(3y^2)^{-3} = \frac{1}{(3y^2)^3} = \frac{1}{3^3 \times y^{2\times 3}} = \frac{1}{27y^6}$

Exercise 2.9

1 Evaluate:

(a) 4^{-1} (b) 3^{-1} (c) 8^{-1} (d) 5^{-3} (e) 6^{-4} (f) 2^{-5}

2 State whether the following are true or false.

(a) $4^{-2} = \frac{1}{16}$ (b) $8^{-2} = \frac{1}{16}$ (c) $x^{-3} = \frac{1}{3x}$ (d) $2x^{-2} = \frac{1}{x}$

3 Write each expression so it has only positive indices.

(a) x^{-2} (b) y^{-3} (c) $(xy)^{-2}$ (d) $2x^{-2}$
(e) $12x^{-3}$ (f) $7y^{-3}$ (g) $8xy^{-3}$ (h) $12x^{-3}y^{-4}$

4 Simplify. Write your answer using only positive indices.

(a) $x^{-3} \times x^4$ (b) $2x^{-3} \times 3x^{-3}$ (c) $4x^3 \div 12x^7$ (d) $\frac{x^{-7}}{x^4}$

(e) $(2x^2)^{-3}$ (f) $(x^{-2})^3$ (g) $\frac{x^{-3}}{x^{-4}}$ (h) $\frac{x^{-2}}{x^3}$

Summary of index laws

$x^m \times x^n = x^{m+n}$	When multiplying terms, add the indices.
$x^m \div x^n = x^{m-n}$	When dividing, subtract the indices.
$(x^m)^n = x^{mn}$	When finding the power of a power, multiply the indices.
$x^0 = 1$	Any value to the power 0 is equal to 1
$x^{-m} = \frac{1}{x^m}$	(when $x \neq 0$).

Fractional indices

The laws of indices also apply when the index is a fraction. Look at these examples carefully to see what fractional indices mean in algebra:

- $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$

  $= x^{\frac{1}{2}+\frac{1}{2}}$ Use the law of indices and add the powers.

  $= x^1$

  $= x$

  In order to understand what $x^{\frac{1}{2}}$ means, ask yourself: what number multiplied by itself will give x?

  $\sqrt{x} \times \sqrt{x} = x$

  So, $x^{\frac{1}{2}} = \sqrt{x}$

- $y^{\frac{1}{3}} \times y^{\frac{1}{3}} \times y^{\frac{1}{3}}$

  $= y^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$ Use the law of indices and add the powers.

  $= y^1$

  $= y$

  What number multiplied by itself and then by itself again will give y?

  $\sqrt[3]{y} \times \sqrt[3]{y} \times \sqrt[3]{y} = y$

  So $y^{\frac{1}{3}} = \sqrt[3]{y}$

This shows that any root of a number can be written using fractional indices. So, $x^{\frac{1}{m}} = \sqrt[m]{x}$.

Worked example 18

1 Rewrite using root signs.

(a) $y^{\frac{1}{2}}$ **(b)** $x^{\frac{1}{5}}$ **(c)** $x^{\frac{1}{y}}$

(a) $y^{\frac{1}{2}} = \sqrt{y}$ **(b)** $x^{\frac{1}{5}} = \sqrt[5]{x}$ **(c)** $x^{\frac{1}{y}} = \sqrt[y]{x}$

2 Write in index notation.

(a) $\sqrt{90}$ **(b)** $\sqrt[3]{64}$ **(c)** $\sqrt[4]{x}$ **(d)** $\sqrt[5]{(x-2)}$

(a) $\sqrt{90} = 90^{\frac{1}{2}}$ **(b)** $\sqrt[3]{64} = 64^{\frac{1}{3}}$ **(c)** $\sqrt[4]{x} = x^{\frac{1}{4}}$ **(d)** $\sqrt[5]{(x-2)} = (x-2)^{\frac{1}{5}}$

A non-unit fraction has a numerator (the number on top) that is not 1. For example, $\frac{2}{3}$ and $\frac{5}{7}$ are non-unit fractions.

Dealing with non-unit fractions

Sometimes you may have to work with indices that are non-unit fractions. For example $x^{\frac{2}{3}}$ or $y^{\frac{3}{4}}$. To find the rule for working with these, you have to think back to the law of indices for raising a power to another power. Look at these examples carefully to see how this works:

$$x^{\frac{2}{3}} = (x^{\frac{1}{3}})^2 \qquad \frac{1}{3} \times 2 \text{ is } \frac{2}{3}$$

$$y^{\frac{3}{4}} = (y^{\frac{1}{4}})^3 \qquad \frac{1}{4} \times 3 = \frac{3}{4}$$

You already know that a unit-fraction gives a root. So we can rewrite these expressions using root signs like this:

$$(x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2 \text{ and } (y^{\frac{1}{4}})^3 = (\sqrt[4]{y})^3$$

So, $(x^{\frac{2}{3}}) = (\sqrt[3]{x})^2$ and $(y^{\frac{3}{4}}) = (\sqrt[4]{y})^3$.

In general terms: $x^{\frac{m}{n}} = x^{m \times \frac{1}{n}} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m$

It is possible that you would want to reverse the order of calculations here and the result will be the same. $x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$, but the former tends to work best.

Worked example 19

Work out the value of:

(a) $27^{\frac{2}{3}}$ **(b)** $25^{1.5}$

(a)	$27^{\frac{2}{3}} = (\sqrt[3]{27})^2$ $= (3)^2$ $= 9$	$\frac{2}{3} = 2 \times \frac{1}{3}$ so you square the cube root of 27.
(b)	$25^{1.5} = 25^{\frac{3}{2}}$ $= (\sqrt{25})^3$ $= (5)^3$ $= 125$	Change the decimal to a vulgar fraction. $\frac{3}{2} = 3 \times \frac{1}{2}$, so you need to cube the square root of 25.

REWIND

You saw in chapter 1 that a 'vulgar' fraction is in the form $\frac{a}{b}$. ◀

Sometimes you are asked to find the value of the power that produces a given result. You have already learned that another word for power is exponent. An equation that requires you to find the exponent is called an *exponential* equation.

Worked example 20

If $2^x = 128$ find the value of x.

$2^x = 128$ $2^7 = 128$ $\therefore x = 7$	Remember this means $2 = \sqrt[x]{128}$. Find the value of x by trial and improvement.